Topology 1 January 7 2014

Lecture 1. Machray 415, TR.
8:30-9:45

- Introduction + Re-schedule!
- Website will have scanned notes that closely follow the book, but perhaps additional examples.
- Textbooks: The one we'll use is Kalajdzievski "Illustrated introduction to topology and homotopy". (Designed for this course).
  Strongly recommend Munkres if you continue in math.
- Marking. 50/50 probably 5-6 assignments.
  Long assignments.

Everyone already saw metric spaces, correct? There we have open sets and closed sets, compact sets, etc, but all properties were based upon the idea of openness of a set. So we formalize this idea.

Definition: A topological space X is a set together with a collection $\mathcal{T}$ of subsets of X, satisfying:

(i) $\emptyset$ and X are in $\mathcal{T}$,

(ii) If $\{U_i\}_{i \in I}$ are in $\mathcal{T}$, so is $\bigcup_{i \in I} U_i$

(iii) If $U_1, \dots, U_n$ are in $\mathcal{T}$, so is $\bigcap_{i=1}^{n} U_n$.

<u>In English</u>: (i) $\mathcal{T}$ contains the empty set and the whole set

(ii) Closed under arbitrary unions,

(iii) Closed under finite intersections.

X <u>nonempty to keep it interesting</u>

<u>Examples</u>:

① Metric spaces are topological spaces.

(i) $\emptyset$ and X are open.

(ii) Unions open

(iii) Intersections are open (take the smallest ball).

② The discrete topology on X is when $\mathcal{T} = P(X)$, the power set. (I.e. every set is open).

③ The indiscrete topology is $\mathcal{T} = \{\emptyset, X\}$.

4) The cofinite topology.

Set $\tau = \{ U \subset X \mid U = \emptyset \text{ or } U^c \text{ is finite} \}$.

We check τ is a topology.

(i) τ contains $\emptyset$ and X. ✓.

(ii) If $\{U_i\}_{i \in I}$ all have finite complements,

then
$$\left(\bigcup_{i \in I} U_i \right)^c \overset{\substack{\text{de} \\ \text{Morgan}}}{=} \underbrace{\bigcap_{i \in I} U_i^c}$$

This is finite, since it is an intersection of finite sets.

So $\bigcup_{i \in I} U_i$ has finite complement, so it's in τ.

(iii) Suppose $U_1, \ldots, U_n$ all have finite complements.

Then
$$\left(\bigcap_{i=1}^{n} U_i \right)^c \overset{\substack{\text{de} \\ \text{Morgan}}}{=} \underbrace{\bigcup_{i=1}^{n} U_i^c}$$

This is a union of finitely many sets having finitely many elts each, so it is finite.

Thus $\bigcap_{i=1}^{n} U_i$ has finite complement, so it's in τ.

Example: Let $X = \mathbb{R}$ and set

$K = \{ \frac{1}{n} \mid n \in \mathbb{Z}, n > 0 \}$. Define τ to be $\emptyset$ with all unions of intervals (a,b) and sets of the form $(a,b) - K$. Then τ is a topology because:

(i) $\emptyset \in \tau$ and $\mathbb{R} = \bigcup_{i=1}^{n} (-n, n) \in \tau$.

(ii) τ consists of unions of (a,b) and $(a,b) - K$, so it's closed under unions.

(iii) Closed under finite intersections.

This will follow from later work on bases of topologies.

Ex: A finite topological space.

Let ~~$X = \{a, 1\}$ and~~

$X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a, b, c\}, \{a\}, \{a, b\}\}$.

(i) Is satisfied.

(ii) We check: ~~pair~~

Since $\emptyset \subset \{a\} \subset \{a, b\} \subset \{a, b, c\}$.

any union is equal to the largest, so it's in τ.

(iii) Any intersection is equal to the smallest, so in τ.

Example 5: The countable complement topology on X.

Set $\tau = \{U \subset X \mid U = \phi \text{ or } U^c \text{ is countable}\}$.

Then τ is a topology, and more or less the same proof works.

Example 6: Let $X = (0,1) \subseteq \mathbb{R}$.

Set $\tau = \{U_n\}_{n \geq 2}$, where $U_n = (0, 1 - \frac{1}{n})$, together with ϕ and X. Then τ is a topology on X:

(i) τ contains ϕ and X by definition.

(ii) A union of U_n's is either a U_n or it is X.

Proof: Consider

$\bigcup_{i \in I} U_i$. If I is finite, then there is a largest i in I. Then $\bigcup_{i \in I} U_i = U_k$, where k is the largest of all numbers in I. Otherwise there is no largest and $\bigcup_{i \in I} U_i = (0,1) = X \in \tau$

(iii) Consider $U_{n_1} \cap \ldots \cap U_{n_k}$. Suppose n_j is the smallest of the n_i's. Then

$$U_{n_1} \cap \ldots \cap U_{n_k} = U_{n_j} \in \tau.$$

Example 7:

Define a topology on the integers $\mathbb{Z}$ as follows.
Let $S(a,b)$ denote the arithmetic sequence

$$S(a,b) = \{an+b \mid n \in \mathbb{Z}\}, \quad a \neq 0.$$

Let τ consist of

- the empty set, and
- all unions of sets of the form $S(a,b)$.

Eg: $S(2,1)$ is $\{\ldots,-5,-3,-1,1,3,5,\ldots\}$

$S(3,2)$ is $\{\ldots,-7,-4,-1,2,5,8,\ldots\}$.

Both are in τ. So is

$S(2,1) \cup S(3,2) = \{\ldots,-7,-5,-4,-3,-1,\ldots \text{etc}\}$.

We check this is a topology.

(i) It contains $\emptyset$ and $\mathbb{Z}$, $\mathbb{Z} = S(1,0)$.

(ii) τ is closed under unions, because if each $\{U_i\}_{i \in I}$ is a union of sets $S(a,b)$, then so is $\bigcup_{i \in I} U_i$.

(iii) Closed under finite intersections is a bit tricky. It boils down to checking that

$$S(a_1, b_1) \cap S(a_2, b_2) \cap \dots \cap S(a_n, b_n)$$

is always in $\mathcal{T}$, but this is exactly the conclusion of the Chinese remainder theorem.

<u>Example 8</u>: <u>The order topology.</u>

Suppose X has a linear order $<$, with smallest ~~ele~~ element $a \in X$ and largest element $b \in X$. Let $\mathcal{T}$ consist of all unions of intervals of the form $[a, x)$, $(y, b]$ and (x, y), where $x, y \in X$. Then this is a topology on X (details to be checked later).

<u>Ex 9</u>: The "usual topology" on $\mathbb{R}$ is actually the order topology.

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Lecture 2.

<u>Recall</u>:

A topology τ on a set X is a collection of subsets (the open sets) of X, satisfying:

(i) $\emptyset$ and X are in τ,

(ii) If $\{U_i\}_{i\in I}$ are in τ, so is $\bigcup_{i\in I} U_i$

(iii) If $U_1, \ldots, U_n$ are in τ, so is $U_1 \cap \ldots \cap U_n$.

(X, τ) together are a <u>topological space</u>.

This is meant to generalized and axiomatize the notion of <u>open sets</u> in metric spaces.

<u>Example</u>: Given X a nonempty set; suppose $\tau = \{\emptyset, X, U_1, U_2, \ldots, U_n, \ldots, \bigcup_{i=1}^{\infty} U_i\}$ satisfying $U_1 \subset U_2 \subset U_3 \subset \ldots$. Then the collection τ is a topology on X, called the <u>nested topology</u>.

Check that τ is a topology:

(i) $\emptyset, X \in \tau$ by def.

(ii) If $\{U_i\}_{i\in I}$ is an arbitrary collection, there are two cases:

a) If I is finite then $\{U_i\}_{i\in I}$ contains a largest set U_k (some k,) and $\bigcup_{i\in I} U_i = U_k \in \mathcal{T}$.

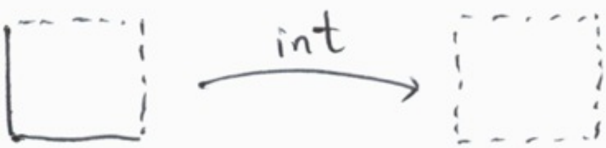
b) If $|I| = \infty$ then $\bigcup_{i\in I} U_i = \bigcup_{k=1}^{\infty} U_k$, which we've included in $\mathcal{T}$.

(iii) If $U_1, \dots, U_n \in \mathcal{T}$, there is a smallest set U_k (some k), and $U_1 \cap \dots \cap U_n = U_k \in \mathcal{T}$.

Thus $\mathcal{T}$ is a topology.

<u>Def</u>: Suppose X is a topological space and $A \subset X$. Then a is an <u>interior point</u> of A if there's an open set U (ie if $\exists\, U \in \mathcal{T}$) such that $a \in U \subset A$.

<u>Example</u>: In the usual (metric) topology on $\mathbb{R}^2$, the interior of $[0,1) \times [0,1)$ is $(0,1) \times (0,1)$



But it's not so clear in general.

<u>Example</u>: Consider $\mathbb{R}$ with the cofinite topology, that is,

$$\mathcal{T} = \{ U \subset \mathbb{R} \mid U^c \text{ is finite or } U = \emptyset \}.$$

Consider the subset (a,b) where a and b are finite.

Then any subset $\mathcal{U} \subset (a,b)$ has infinite complement, so cannot be open. Since (a,b) contains no open sets, it has no interior points.

<u>Proposition</u>: A subset A of a topological space X is open iff every $a \in A$ is an interior point.

<u>Proof</u>: $(\Rightarrow)$ Obvious.

$(\Leftarrow)$ Suppose every point of a is an interior point. Then for each a, there's an open set $\mathcal{U}_a$ with $a \in \mathcal{U}_a \subset A$. But then $A = \bigcup_{a \in A} \mathcal{U}_a$ is a union of open sets, and so is open.

<u>Definition</u>: The set of interior points of A is called the <u>interior of A</u>, and is written $\operatorname{int} A$.

<u>Proposition</u>: If $\mathcal{U} \subset A$ and $\mathcal{U}$ is open, then $\mathcal{U} \subset \operatorname{int} A$. (ie ~~$\mathcal{U}$~~ $\operatorname{int}A$ is the largest open subset of A).

<u>Proof</u>: If $\mathcal{U} \subset A$ and $\mathcal{U}$ is open, then every point of $\mathcal{U}$ is an interior point and $\mathcal{U} \subset \operatorname{int} A$.

<u>Example</u>:

Consider $(0,1) \subseteq \mathbb{R}$ with the topology
$\tau = \{(0, 1-\tfrac{1}{n}) \mid n \in \mathbb{Z}, n > 0\}$.

Then $\operatorname{int}(0, 5/7)$ is the largest set of the form $(0, 1-\frac{1}{n})$ contained in $(0, 5/7)$, so it's $(0, 2/3)$. (Check $\frac{2}{3} < \frac{5}{7} < \frac{3}{4}$).

Example: If $\mathbb{R}$ is given the cofinite topology, then

$\operatorname{int} A = A$ if A^c is finite, and

$\operatorname{int} A = \emptyset$ if A^c is infinite, because in this case A cannot contain an open set.

Definition: A subset A of a topological space X is closed if $A \setminus X$ is open.

From the definition of a topology τ on X, we get:

Theorem: Suppose (X, τ) is a topological space. Then:

(i) $\emptyset, X$ are closed.

(ii) If $\{U_i\}_{i \in I}$ are closed, so is $\bigcap_{i \in I} U_i$.

(iii) If $U_1, \ldots, U_n$ are closed, so is $U_1 \cup \ldots \cup U_n$.

Proof: De Morgan's Laws.

Remarks: Sets both open and closed (e.g $\emptyset, X$) are sometimes called clopen (ugh).

Example: Consider $\{\frac{1}{n} \in \mathbb{R} \mid n \in \mathbb{Z}, n > 0\} = K$.

In the usual topology on $\mathbb{R}$, the set K is not ~~op~~ closed because the complement is not open.

On the other hand, if we define a topology τ on $\mathbb{R}$ to ~~be~~ consist of:

- intervals $(a,b) \subset \mathbb{R}$
- sets $(a,b) - K$
- all unions of these types of sets.

(called the K-topology on $\mathbb{R}$).

Then

$$K^c = \bigcup_{i=1}^{\infty} \left((-i, i) - K\right),$$ which is open. So K is closed.

Def: An open nbhd of $x \in X$ is an open subset $\mathcal{U}$ with $x \in \mathcal{U}$.

The point x is an accumulation point of A if every open neighbourhood of x contains points of A.

Theorem: A subset $A \subset X$ is closed iff it contains all of its accumulation points.

Proof: ($\Rightarrow$) Suppose x is an accumulation point of A, and $x \notin A$. Then $x \in A^c$ and x is not an interior point, so A^c is not open. Therefore A is not closed.

($\Leftarrow$) Suppose A contains all its interior points. Then $\forall x \in A^c$, $\exists\ \mathcal{U}$ s.t. $x \in \mathcal{U}$ and $\mathcal{U} \cap A = \emptyset$, i.e. $\mathcal{U} \subset A^c$. Therefore $\forall x \in A^c$, x is an interior point of A^c, i.e. A^c is open. So A is closed.

<u>Notation</u>: The set of all accumulation points of A will be written A'.

<u>Example</u>: In the K-topology on $\mathbb{R}$, 0 is <u>not</u> an accumulation point of K, because e.g. $0 \in (-1,1) - K$ is open an contains no points of K. So in $\mathbb{R}$ with the K-topology, $K' = K$.

For the next theorem (a named one) we need a lemma.

<u>Lemma</u> (Cantor's nested intervals theorem)
Let $I_1 \supset I_2 \supset I_3 \supset \ldots$ be a set of nested, closed intervals in $\mathbb{R}$. Then $\bigcap_{k=1}^{\infty} I_k \neq \emptyset$.

(<u>Proof</u>: 2nd yr analysis type question).

Recall that a subset of a metric space is <u>bounded</u> if it is contained in some closed ball $B(x,\varepsilon)$

Theorem (Bolzano-Weierstrass).

Every bounded, infinite subset of $\mathbb{R}$ has an accumulation point.

Proof: WLOG assume $A \subset [0,1] = I_1$

Of the intervals $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$, choose ~~the~~ one containing infinitely many points of A and call that interval I_2. In general, to create I_{n+1} you cut I_n into (closed) halves, and choose ~~the~~ a half with infinitely many points from A in it.

Then from Cantor's lemma, $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$. Choose ~~a∈In~~ $a \in \bigcap_{n=1}^{\infty} I_n$.

Then choose a nbhd of a, say $(a-\varepsilon, a+\varepsilon)$.

By construction there is a large enough j s.t. $I_j \subset (a-\varepsilon, a+\varepsilon)$, so $(a-\varepsilon, a+\varepsilon)$ contains infinitely many points of A since I_j does.

Thus every nbhd of a contains points of A other than a itself, so $a \in A'$. (ie. we found an acc. pt).

This will be generalized to a top. space in 7.4!

Def: The closure of A is the smallest closed subset containing A, and is denoted $\bar{A}$.
(ie. if $A \subset U$ and U is closed, then $U \subset \bar{A}$).

<u>Theorem</u>: Let $A, B \subset X$.

(a) If $A \subset B$ then $A' \subset B'$

(b) $\bar{A} = A \cup A'$.

<u>Proof</u>: .... Left to the student to read book.

<u>Example</u>: