ORDERABLE GROUPS AND TOPOLOGY MINICOURSE NOTES

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Abstract. The goal of this minicourse is to study the orderability properties of fundamental groups of 3-manifolds, and when possible, explain orderability or non-orderability of the fundamental group via topological properties of the manifold. In particular it covers bi-orderability of knot groups, connections with foliations, group actions and the L-space conjecture; these notes include plenty of open problems and conjectures that are active areas of research.

Goal: To study orderability properties of fundamental groups of 3-manifolds M, specifically which fundamental groups are left orderable (and which are not), which are bi-orderable (and which are not). When possible, explain orderability or non-orderability of $\pi_1(M)$ via topological properties of M.

Owing to a theorem of Boyer, Rolfsen and Wiest (covered in the first section), this material is naturally best organized into two cases: The case of infinite first homology, and the case when the first homology is finite, so the material will be organized as follows:

Lecture 1: Manifolds with $|H_1(M)| = \infty$
Lecture 2: Manifolds with $|H_1(M)| < \infty$: Seifert fibred manifolds
Lecture 3: Manifolds with $|H_1(M)| < \infty$: L-spaces and Dehn fillings

1. Manifolds with $|H_1(M)| = \infty$

Let M be a compact, connected, orientable 3-manifold. We stick to orientable manifolds here only to make the discussion easier, to deal with non-orientable cases one needs only to do a bit of extra work (or consult the references). For background on 3-manifolds, see [45].

We first reduce the question of orderability of $\pi_1(M)$, where M is arbitrary, to a class of simpler manifolds. For this we need a definition.

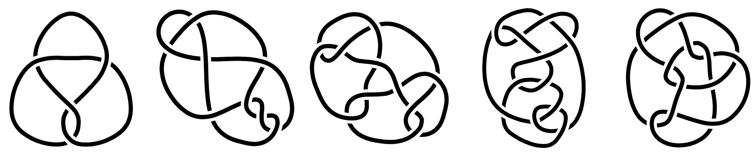
Definition 1.1. *A 3-manifold M is irreducible if every tamely embedded 2-sphere in M bounds a ball.*

Example 1.2. *The sphere S^3 is irreducible, as are lens spaces*

$$L(p,q) = S^3/\sim, \text{ where } (z_1, z_2) \sim (e^{\frac{2\pi i}{p}} \cdot z_1, e^{\frac{2p\pi i}{q}} \cdot z_2)$$

Date: September 7, 2014.

where p, q are coprime with $p \neq 0$, as are complements of nontrivial knots in S^3:



Manifolds which are not irreducible are $S^2 \times S^1$, or manifolds obtained by gluing together two manifolds $M_i \neq B^3$ $(i = 1, 2)$ by cutting a copy of B^3 out of each and gluing along the resulting boundaries. The resulting manifold is $M_1 \sharp M_2$.

The prime decomposition theorem says that every orientable 3-manifold M admits a decomposition into irreducible 3-manifolds M_i and copies of $S^2 \times S^1$ (these are prime manifolds):

$$M = M_1 \sharp \ldots \sharp M_k \sharp (S^2 \times S^1) \sharp \ldots \sharp (S^2 \times S^1)$$

As such, for every orientable 3-manifold its fundamental group is a free product

$$\pi_1(M) = \pi_1(M_1) * \ldots * \pi_1(M_k) * \mathbb{Z} * \ldots * \mathbb{Z}.$$

where the M_i's are irreducible. Now we can reduce the problem of left- or bi-orderability to a simpler problem, thanks to a theorem of Vinogradov:

Theorem 1.3 (Vinogradov, [48])**.** *The free product $G_1 * \ldots * G_k$ of groups is left-orderable (resp. locally indicable or bi-orderable) if and only each factor is left-orderable (resp. locally indicable or bi-orderable).*

So, in order to understand orderability of fundamental groups of 3-manifolds, it suffices to consider orderability of $\pi_1(M)$ where M is irreducible.

1.1. Left-ordering fundamental groups when $|H_1(M)| = \infty$.

With this restriction, one could say that the 'fundamental theorem' for left-ordering 3-manifold groups is

Theorem 1.4 (Boyer-Rolfsen-Wiest [3], Howie-Short [26])**.** *If M is a compact, connected, orientable, irreducible 3-manifold ($M \neq S^3$) then $\pi_1(M)$ is left-orderable if and only if there exists a surjection $\pi_1(M) \to L$ onto a nontrivial left-orderable group.*

It is then clear why one might naturally want to cover the case $|H_1(M)| = \infty$ first, because this restriction gives $\pi_1(M)$ a left-orderable quotient.

Corollary 1.5. *If M is a compact, connected, orientable, irreducible 3-manifold and $|H_1(M)| = \infty$ then $\pi_1(M)$ is left-orderable.*

Proof. If $|H_1(M)| = \infty$ then the fundamental group has $\mathbb{Z}$ as a quotient. □

Example 1.6. *All knot groups are left-orderable, since $H_1(S^3 \setminus K) = \mathbb{Z}$ for all knots $K \subset S^3$.*

More generally, if M is compact connected, orientable, irreducible and $\partial M \neq 0$ contains no copies of S^2, then an Euler characteristic argument gives $|H_1(M)| = \infty$ and so $\pi_1(M)$ is left-orderable.

In order to prove Theorem 1.4, there are a couple of theorems that one must have in hand–one from the theory of orderable groups, and one from 3-manifold topology.

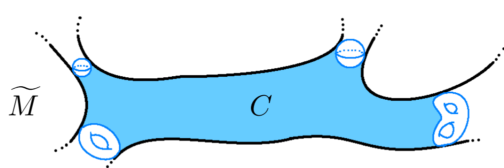
Theorem 1.7 (Burns-Hale [5])**.** *A group G is left-orderable if and only if for every finitely generated nontrivial $H \subset G$ there exists a surjection $H \to L$, where L is a nontrivial left-orderable group.*

Theorem 1.8 (Scott [46])**.** *Suppose that M is a noncompact 3-manifold and that $\pi_1(M)$ is finitely generated. Then there exists a compact submanifold $C \subset M$ such that the inclusion $i : C \to M$ induces an isomorphism $i_* : \pi_1(C) \to \pi_1(M)$.*

Proof of Theorem 1.4, [3]. One direction is trivial, the difficult direction is to show that a nontrivial homomorphism $\pi_1(M) \to L$ onto a left-orderable group is sufficient to left-order $\pi_1(M)$. We show this by applying Burns-Hale, so let $H \subset \pi_1(M)$ be finitely generated and nontrivial.

Case 1: $|\pi_1(M) : H| < \infty$. Then we have a nontrivial map $H \to L$ by restriction.

Case 2: $|\pi_1(M) : H| = \infty$. Then there exists a covering space $p : \widetilde{M} \to M$ such that $p_*(\pi_1(\widetilde{M})) = H$. While $\widetilde{M}$ is noncompact, its fundamental group is finitely generated by assumption. So Scott's theorem gives us a compact core $C \subset \widetilde{M}$ such that $i_* : \pi_1(C) \to \pi_1(\widetilde{M})$ is an isomorphism.

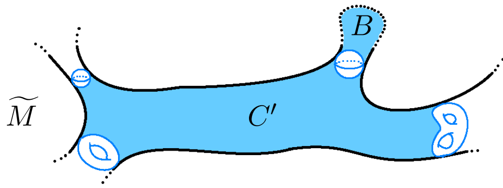


Note that we must have $\partial C \neq \emptyset$, and in fact we can be sure that ∂C contains no 2-spheres as follows:

If ∂C did contain 2-spheres, then $S^2 \subset \widetilde{M}$ would bound a ball, since irreducibility is inherited by coverings (see [24], for example). So there exists $B \subset \widetilde{M}$ satisfying one of the following two cases:

(i) $C \subset B$, which is impossible since $i_* : \pi_1(C) \to \pi_1(\widetilde{M})$ is nontrivial.

(ii) $B \cap \operatorname{int}(C) \neq \emptyset$, in which case $C' = B \cup C$ serves as a new "compact core" for $\widetilde{M}$, but with one less sphere in its boundary. So, we can fill all the 2-spheres.



But now if ∂C contains no 2-spheres (and is still nonempty!), then $|H_1(C)| = \infty$. Thus there exists a map $H \cong \pi_1(C) \to H_1(C) \to \mathbb{Z}$, and Burns-Hale shows that $\pi_1(M)$ must be left-orderable.

□

We can actually glean more from this proof than we had first hoped. Note that for every finitely generated infinite index subgroup H, we actually showed that there exists a homomorphism $H \to \mathbb{Z}$. Therefore, as long as every finitely generated finite index subgroup has $\mathbb{Z}$ as a quotient, we've shown that the group is actually locally indicable. Every finite index subgroup will have $\mathbb{Z}$ as a quotient as long as the group itself does, and so we have

Corollary 1.9. *If M is compact, connected, orientable and irreducible and $|H_1(M)| = \infty$, then $\pi_1(M)$ is locally indicable.*

Thus, in the case $|H_1(M)| = \infty$ the only "orderability property" that remains to be investigated is bi-orderablity of $\pi_1(M)$.

1.2. **Bi-ordering fundamental groups when $|H_1(M)| = \infty$.** First we note that the restriction of $|H_1(M)| = \infty$ is a necessary restriction if we are to talk about bi-orderability. We're assuming that M is compact, and therefore $\pi_1(M)$ is finitely generated, say by $g_1, \ldots, g_k$. Without loss of generality say g_k is the largest of the generators–then there exists a largest convex subgroup C not containing g_k. Moreover since this is a bi-ordering, the largest convex subgroup is normal and $\pi_1(M)/C$ inherits and Archimedean ordering, and is therefore abelian (and obviously infinite). But then the map $\pi_1(M) \to \pi_1(M)/C$ must factor through the abelianization $H_1(M)$, so $H_1(M)$ must be infinite (see [16] for details of this argument in the case of Conradian orders, which apply in the bi-orderable case as well).

Also of note is that the question of *virtual* bi-orderability has already been largely resolved. It is a consequence of Ian Agol's proof of the virtual Haken conjecture that every $\pi_1(M)$, when M is closed and hyperbolic, is *virtually* bi-orderable [1]. This follows from showing that every M has a finite sheeted cover whose fundamental group is isomorphic to a subgroup of a RAAG, which is a bi-orderable group. Wise and Przytycki have also recently dealt with the case of graph manifolds [44]. So while the question of virtual bi-orderability is interesting, the primary question that remains is whether or not $\pi_1(M)$ itself is bi-orderable (and not some finite-index subgroup).

In general, this seems to be a difficult problem and so results thusfar are piecemeal. The manifolds that have been considered to date are Seifert fibered manifolds, knot complements, some 3-manifolds fibering over S^1, and manifolds whose fundamental groups have two generators and one relator. To keep things simple, we'll focus on knot complements.

Recall a knot K is said to be fibered if its complement fibers over S^1 with fiber a surface Σ with $\partial\Sigma \neq \emptyset$. The first theorem we have is:

Theorem 1.10. *Suppose that $K \subset S^3$ is fibered, and let $\Delta_K(t)$ denote the Alexander polynomial of K. Then:*

(1) *If all the roots of $\Delta_K(t)$ are positive real numbers, then $\pi_1(S^3 \setminus K)$ is bi-orderable* [43] *.*
(2) *If $\pi_1(S^3 \setminus K)$ is bi-orderable, then $\Delta_K(t)$ has at least one positive real root.* [13].

There are some generalizations of these theorems, but again the results only apply in the case where the manifold fibers. For example, the same theorems hold if you replace the fibered knot complement $S^3 \setminus K$ with the mapping torus

$$M \cong \Sigma \times [0,1]/\sim \text{ where } (x,0) \sim (\phi(x),1)$$

for some $\phi : \Sigma \to \Sigma$. In this case we interpret the Alexander polynomial as the characteristic polynomial of $\phi_* : H_1(\Sigma) \to H_1(\Sigma)$. The proof of this fact is more or less the same as the proof of the knot complement case, but some easy results about free groups need to be replaced with more difficult theorems about surface groups [33].

It is also possible to slightly relax the condition of all roots of $\Delta_K(t)$ being positive in the statement of (1) above. In [34], the authors show that (1) holds if you replace "$\Delta_K(t)$ has all roots positive" with "$\Delta_K(t)$ is special", where special is a restrictive condition defined for the purpose of their paper. Special polynomials need not have all positive roots, yet a special polynomial still satisfies some necessary properties that make the proof of (1) possible.

There have also recently been some advances that allow for the investigation of non-fibered knots. To state these new theorems, we need to prepare some technical combinatorial definitions having to do with the relators of a group presentation.

We write a^b in place of $b^{-1}ab$, and for a word $w \in F(a,b)$ in the free group on generators a and b, write w_b and w_a for the total exponent sum of b and a in the word w. If $w_b = 0$ then we can rewrite w as

$$w = a^{m_1 b^{d_1}} \cdots a^{m_r b^{d_r}}$$

for some integers m_i, d_i and $r \geq 1$. For all $j \in \mathbb{Z}$, set $\tau_j(w) = \{i : d_i = j\}$ and let $S_w = \{j : \sum_{i \in \tau_j(w)} m_i \neq 0\}$.

We say that a word w of the form above is *tidy* if $\tau_j(w) = \emptyset$ for all j satisfying either $j > \max\{S_w\}$ or $j < \min\{S_w\}$. Set $\ell = \max\{S_w\}$; we say that w is *principal* if $|\tau_\ell(w)| = 1$. In the case that w is principal and

$\tau_\ell(w) = \{k\}$, we call w *monic* if in addition $m_k = 1$. Set $s = \min\{S_w\}$, when $\pi_1(S^3 \setminus K) = \langle a, b|w\rangle$ with w as above, the Alexander polynomial has formula $\Delta_K(t) = \sum_{i=1}^{r} m_i t^{d_i - s}$. We can group like powers and rewrite this as

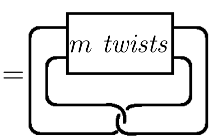
$$\Delta_K(t) = \sum_{j\in\mathbb{Z}} \left(\sum_{i\in\tau_j(w)} m_i \right) t^{j-s}$$

where we understand that the coefficient of t^{j-s} is zero when $\tau_j(w) = \emptyset$.

Theorem 1.11 (Chiswell-Glass-Wilson, [10])**.** *Let K be a knot in S^3, and suppose that $\pi_1(S^3 \setminus K)$ has a presentation of the form $\langle a, b|w\rangle$ where w is tidy. Let $\Delta_K(t)$ denote the Alexander polynomial of K. Then:*

1. *If $\pi_1(S^3 \setminus K)$ is bi-orderable, then $\Delta_K(t)$ has a positive real root.*
2. *If w is monic and all the roots of $\Delta_K(t)$ are real and positive, then $\pi_1(S^3 \setminus K)$ is bi-orderable.*
3. *If w is principal, $\Delta_K(t) = a_0+\cdots+a_{d-1}t^{d-1}-mt^d$ where $\gcd\{a_0, \ldots, a_{d-1}\} = 1$ and a_{d-1} is not divisible by m, and all the roots of $\Delta_K(t)$ are real and positive, then $\pi_1(S^3 \setminus K)$ is bi-orderable.*

Example 1.12 (C-Desmarais-Naylor, [12])**.** *Consider the twist knots with m twists:*

$K_m =$ 

When $m \geq 2$ is even,

$$\Delta_{K_m}(t) = -\frac{m}{2} + (m+1)t - \left(\frac{m}{2}\right) t^2$$

and both roots are real. If $m = 2$ then K_m is the figure eight knot, which fibres, so the group is bi-orderable by [43]*. For $m > 2$ it does not fiber, and you need to use the presentation*

$$\pi_1(S^3 \setminus K_m) = \langle x, y|(y^{-1}xy)(x^{-\frac{m}{2}})(y^{-1}x^{\frac{m}{2}}y)(y^{-2}x^{-\frac{m}{2}}y^2)(y^{-1}x^{\frac{m}{2}}y)\rangle$$

and check that the relator satisfies the technical condition of being "principal" (which it does), and that the coefficients of the Alexander polynomial satisfy certain divisibility conditions (they do). So again, the group is bi-orderable, this time by [10]*.*

On the other hand when $m \geq 3$ is odd,

$$\Delta_{K_m}(t) = \frac{m+1}{2} - mt + \left(\frac{m+1}{2}\right) t^2$$

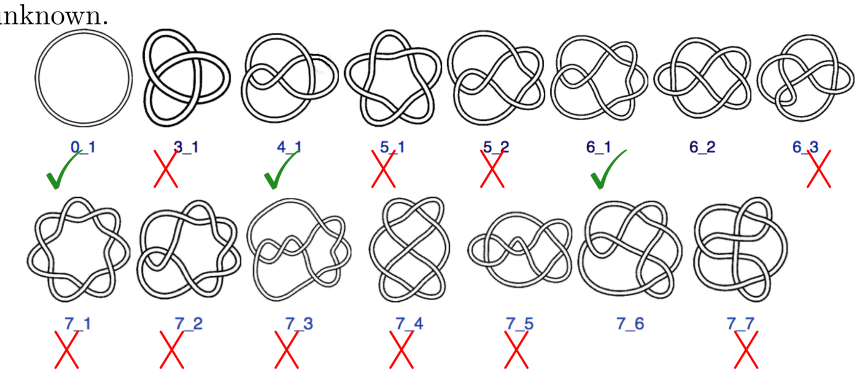
both roots are imaginary and K_m does not fiber. The presentation in this case is

$$\pi_1(S^3 \setminus K_m) = \langle x, y|(y^{-1}xy)(x^{-\frac{m-1}{2}})(y^{-1}x^{\frac{m+1}{2}}y)(y^{-2}x^{-\frac{m+1}{2}}y^2)(y^{-1}x^{\frac{m-1}{2}}y)\rangle$$

which is a presentation with a "tidy" word, so we apply [10] *to conclude the group is not bi-orderable.*

These are all the available theorems that I know of which have been applied to knot groups. If we plug away at the knot tables to see how many knot groups we can understand using these theorems, we do pretty well at the beginning:

Here, we write ✓ for biorderable, ✗ for not bi-orderable, and blank for unknown.



For higher crossing number the results are more sparse. For example if we consider 12 crossing knots, we can only determine orderability of 27 of the first 100 knot groups coming from knots with 12 crossings [12]. In general, the 'percentage' of knots that we can deal with becomes very small as the crossing number becomes large.

All of the availble theorems about bi-orderability are proved using a similar argument (though with substantially different details). So, as an example of how the proofs go, let's consider a proof of Theorem 1.10(1) from [43].

So we prove: If K is fibered and $\Delta_K(t)$ has all positive roots, then $\pi_1(S^3 \setminus K)$ is bi-orderable.

Here, "K fibered" means that $S^3 \setminus K$ is a mapping cylinder

$$S^3 \setminus K \cong (\Sigma \times [0,1])/\sim \text{ where } (x,1) \sim (h(x),0)$$

for some surface with boundary Σ and some map $h : \Sigma \to \Sigma$ called the monodromy. For a fibered knot, $\Delta_K(t)$ is the characteristic polynomial of the linear map $H_1(\Sigma) \to H_1(\Sigma)$ induced by h.

We can compute the fundamental group, it turns out to be the HNN extension

$$\pi_1(S^3 \setminus K) = \langle x_1, \ldots, x_{2g}, t | h_*(x_i) = tx_it^{-1}\rangle$$

where $h_* : \pi_1(\Sigma) \to \pi_1(\Sigma)$ is the induced map.

So we have a surjection onto the infinite cyclic group $\langle t\rangle$

$$1 \to \pi_1(\Sigma) \to \pi_1(S^3 \setminus K) \to \langle t\rangle \to 1$$

where, since Σ is compact and has boundary, $\pi_1(\Sigma)$ is a finitely generated free group. Therefore $\pi_1(S^3 \setminus K)$ will be bi-orderable if (exercise: and only if) $\pi_1(\Sigma)$ has an ordering preserved by conjugation by t, i.e. by $h_* : \pi_1(\Sigma) \to \pi_1(\Sigma)$. So the lemma we need is:

Lemma 1.13 (Perron-Rolfsen, [43])**.** *Let F be a finitely generated free group and $h : F \to F$ an automorphism. Suppose that all the eigenvalues of $h_* : H_1(F;\mathbb{Q}) \to H_1(F;\mathbb{Q})$ are real and positive. Then there exists a bi-ordering of F preserved by h.*

We prove the lemma by constructing an invariant bi-ordering in a classical way, using the lower central series quotients F_i/F_{i+1}: You can order F by choosing orderings $<_i$ of each F_i/F_{i+1} and declaring $x > 1$ if $xF_{i+1} \in F_i/F_{i+1}$ is positive, where i is the largest subscript such that $x_i \in F_i$. We just need to take some care in choosing $<_i$ so that the result is invariant under h.

First, if a linear map $L : \mathbb{Q}^n \to \mathbb{Q}^n$ has all positive eigenvalues, then it preserves a bi-ordering of $\mathbb{Q}^n$ (think lexicographically, though this claim takes some work).

Then the map $h : F \to F$ induces maps

$$h_i : F_i/F_{i+1} \to F_i/F_{i+1}$$

each of which can be shown to be a restriction to a certain subspace of the map

$$h^{\otimes i} : (F/F')^{\otimes i} \to (F/F')^{\otimes i}$$

The assumption that h_* has all positive eigenvalues means the same is true of every $h^{\otimes i}$, since the eigenvalues of $h^{\otimes i}$ are i-fold products of eigenvalues of h.

So, every $h^{\otimes i}$ preserves an ordering, and thus so does every $h_i : F_i/F_{i+1} \to F_i/F_{i+1}$, call the preserved ordering $<_i$. Now use the "classical construction" above, and get an ordering of F preserved by $h : F \to F$.

1.3. **Questions.** There are several natural questions that arise as a result of how 'sparse' our understanding of bi-orderability is:

Q1. Is the knot group of 6_2 bi-orderable?

Q2. If $\pi_1(S^3 \setminus K)$ is bi-orderable, must $\Delta_K(t)$ have a positive real root? Does one positive root imply that the group if bi-orderable? How many positive roots do you need?

Remarks. An example of Chiswell-Glass-Wilson appearing in [10] shows that there exist HNN extensions $\langle a, b | r \rangle$, for which we can define an "Alexander polynomial" which satisfies:

- $\langle a, b | r \rangle$ is not bi-orderable, and
- there exists a positive real root.

However their groups do not arise as knot groups.

Q3.[10] Call and element $g \in G$ generalized torsion if there exist $h_i \in G$ such that

$$\prod_{i=1}^{n} h_i g h_i^{-1} = 1$$

If there is no generalized torsion in a knot group, does it imply that the knot group is bi-orderable?

Remarks. Dale Rolfsen and his student Geoff Naylor have shown that knot groups admit generalized torsion. This is an easy fact for torus knot groups, but for hyperbolic knots finding generalized torsion (if it exists) seems harder in general. They computed, for example, that the knot group of 5_2 is

$$G = \langle a, b : b^2A^2b^2aB^3a \rangle$$

(where we use A and B in place of a^{-1} and b^{-1} to simplify notation) and then with $w = AbaB$ they found

$$(a^4B^2wb^2A^4)(a^4BaB^2wb^2abA^4)(a^4BaBwbAbA^4)(bawAB)(bwB)(b^2wB^2) = 1.$$

Therefore w is generalized torsion.

Q4. [34] If F is a free group, what are necessary and sufficient conditions for an automorphism $\phi : F \to F$ to preserve a bi-ordering?

Remarks. This is precisely what is needed to bi-order the groups G that arise from short exact sequences

$$1 \to F \to G \to \mathbb{Z} \to 1$$

for example, fibered knot groups. There is some work towards answering this, see [34]. What about conditions for some power $\phi^k : F \to F$ to preserve a bi-ordering, so we can find a finite-index bi-orderable group corresponding to the cover with monodromy ϕ^k?

Q5. Is there a topological meaning to bi-orderability of $\pi_1(M)$, aside from Alexander polynomial restrictions?

2. Manifolds with $|H_1(M)| < \infty$: The Seifert fibered case

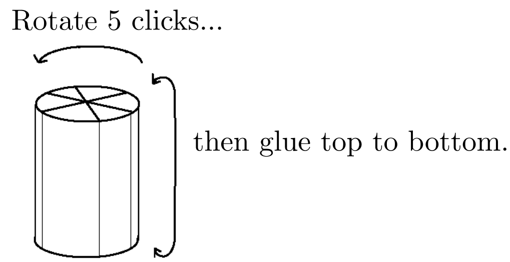
In this section, since we are restricting to finite first homology none of the groups under consideration can be either bi-orderable or locally indicable, as each implies $|H_1(M)| = \infty$. So, the problem we'll consider is when $\pi_1(M)$ is left-orderable.

One of our stated goals was also to explain *why* $\pi_1(M)$ is left-orderable based upon the topology of M. There is a very satisfying answer to this in the case that M is a Seifert fibered manifold, so we restrict to Seifert fibred manifolds for now and explain the topological meaning of left-orderings of $\pi_1(M)$.

First, we introduce Seifert fibered manifolds following Hatcher [24], for more details see [29]. A *model Seifert fibering* of the solid torus $D^2 \times S^1$ is a decomposition of $D^2 \times S^1$ into disjoint circles, called fibers. The fibers are constructed by taking the solid torus T^2 and building it as

$$T^2 = (D^2 \times [0, 1])/\sim$$

where $\sim$ identifies $D^2 \times \{0\}$ with $D^2 \times \{1\}$ with a $\frac{2\pi p}{q}$ twist for some $\frac{p}{q} \in \mathbb{Q}$ (p, q, relatively prime). Illustrated below is the case $q = 6$, then if $p = 5$, for example, we would rotate the top by 5 'clicks' and glue it to the bottom.



The fibers of T^2 are then built out of segments $\{x\} \times [0, 1]$ and are of two kinds:

- The image of $\{0\} \times [0, 1]$, called an exceptional fiber
- The image of q equally spaced segments $\{x\} \times [0, 1]$ which are glued together end to end, called a regular fiber.

A *Seifert fibering* of a 3-manifold is a decomposition of M into circles such that every circle (fiber) has a neighbourhood that is fiber-preserving diffeomorphic to the neighbourhood of a fiber in a fibered solid torus. Such a manifold is called a Seifert fibered manifold.

Here is a way to construct Seifert fibered manifolds. Let Σ be a compact, connected surface with b boundary components. Choose disks $D_1, \dots, D_n \subset \text{int}(\Sigma)$, and let

$$\Sigma' = \Sigma \setminus (\text{int}(D_1) \cup \ldots \cup \text{int}(D_n))$$

Let $M' \to M$ be an oriented S^1-bundle over Σ', so if Σ' is orientable then $M' \cong S^1 \times \Sigma$. Otherwise it is a bundle with a torus over every orientation-preserving path in Σ, and a Klein bottle over the orientation-reversing ones.

Note that above each $\partial D_i \subset \Sigma'$ there's a corresponding $T_i \subset \partial M'$. Fix a section $s : \Sigma' \to \Sigma$, which fixes a basis $\pi_1(T_i)$ for each i, namely our basis will be $[h_i^*] = [s(\partial D_i)]$ and $[h_i]$, the class of the fiber $S^1 \hookrightarrow M'$. Now with a fixed basis, we have a correspondence between curves $\gamma : S^1 \hookrightarrow T_i$ and fractions $\frac{\beta_i}{\alpha_i} \in \mathbb{Q} \cup \{\infty\}$ by representing the class of such curves relative to our basis:

$$[\gamma] = \alpha_i[h_i^*] + \beta_i[h_i]$$

We can construct a Seifert fibered manifold M over the surface Σ by choosing fractions $\frac{\beta_i}{\alpha_i} \subset \mathbb{Q}$, for $i = 1, \dots, n$. Then to each $T_i \subset \partial M'$, attach $D^2 \times S^1$ by gluing $\partial D^2 \times \{y\}$ to the curve $\alpha_i[h_i^*] + \beta_i[h_i]$ on T_i. Writing g for the genus of Σ and b for the number of boundary components, we denote the resulting manifold by

$$M(\pm g, b; \frac{\beta_1}{\alpha_1}, \dots, \frac{\beta_n}{\alpha_n})$$

with $\pm$ in front of the g to indicate whether or not Σ is orientable.

This is a good way to understand Seifert fibered manifolds, since

Proposition 2.1. *Every orientable Seifert fibered manifold is fiber-preserving diffeomorphic to one of the model manifolds* $M(\pm g, b; \frac{\beta_1}{\alpha_1}, \dots, \frac{\beta_n}{\alpha_n})$ *for some choice of* Σ *and fractions* $\frac{\beta_i}{\alpha_i}$. *Two models are diffeomorphic if and only if* $\frac{\beta_i}{\alpha_i} \cong \frac{\beta_i'}{\alpha_i'}$ *mod* 1 *(up to permuting indices) and if* $b = 0$ *then also* $\Sigma \frac{\beta_i}{\alpha_i} = \Sigma \frac{\beta_i'}{\alpha_i'}$.

Shortening the names of these manifolds to simply 'M' and then using the Siefert Van Kampen theorem, we can calculate that the fundamental groups under consideration in this section are:

$$\pi_1(M) = \langle a_1, b_1, \dots, a_g, b_g, \gamma_1, \dots, \gamma_n, h | \\ h \text{ central }, \gamma_j^{\alpha_j} = h^{-\beta_j}, [a_1, b_1] \dots [a_n, b_n]\gamma_1 \dots \gamma_n = h^b \rangle$$

if $g \geq 0$ and

$$\pi_1(M) = \langle a_1, \dots, a_{|g|}, \gamma_1, \dots, \gamma_n, h | \\ a_j h a_j^{-1} = h^{-1}, \gamma_j^{\alpha_j} = h^{-\beta_j}, \gamma_j h \gamma_j^{-1} = h, a_1^2 \dots a_{|g|}^2 \gamma_1 \dots \gamma_n = h^b \rangle$$

if $g < 0$. In the case that either $g \neq 0, -1$ or the boundary of Σ is nonempty, these groups have infinite abelianization, i.e. $|H_1(M)| = \infty$. These cases were dealt with in the last section, and we find

Theorem 2.2. [3] *If* M *is Seifert fibered and* $|H_1(M)| = \infty$, *then* $\pi_1(M)$ *is left-orderable as long as* $M \ncong \mathbb{R}P^2 \times S^1$.

Proof. Use Theorem 1.4, and deal with the few reducible cases by hand. □

So the question of left-orderability is, as expected, only interesting the case $|H_1(M)| < \infty$. In this case Σ is either S^2 or $\mathbb{R}P^2$ and the groups in question become

$$\pi_1(M) = \langle \gamma_1, \dots, \gamma_n, h | h \text{ central }, \gamma_j^{\alpha_j} = h^{-\beta_j}, \gamma_1 \dots \gamma_n = 1 \rangle$$

and

$$\pi_1(M) = \langle \gamma_1, \dots, \gamma_n, y, h | \\ y h y^{-1} = h^{-1}, \gamma_j^{\alpha_j} = h^{-\beta_j}, \gamma_j h \gamma_j^{-1} = h, y^2 \gamma_1 \dots \gamma_n = 1 \rangle$$

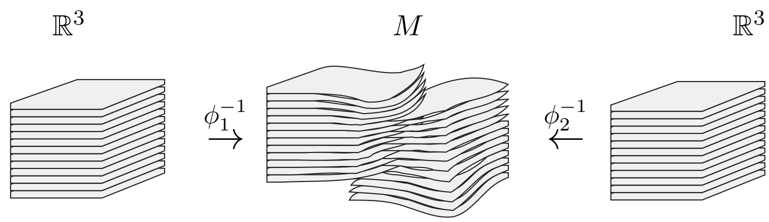
We have a theorem that characterizes orderability completely in terms of the topology of M.

Theorem 2.3. [3] *If* M *is Seifert fibered and* $|H_1(M)| < \infty$ *then* $\pi_1(M)$ *is left-orderable if and only if* M *admits a co-orientable horizontal foliation.*

First, we digress for a moment to introduce foliations, for more information see [8, 9]. Here, our foliations will be codimension one, meaning that we can partition the manifold M into connected surfaces called leaves, and cover M by charts $\phi : U \to \mathbb{R}^2 \times \mathbb{R}$ such that each leaf $L \subset M$ satisfies

$$\phi(L \cap U) = \bigcup_{i \in I} \mathbb{R}^2 \times \{y_i\}$$

In other words, each connected component of $L \cap U$ is a small bit of a plane, called a plaque. The plaques piece together to form leaves. This can also be turned around and understood by saying that the atlas $\{U_\alpha, \phi_\alpha\}$ of M has charts so that the planes fit together neatly on M, as in



We are considering foliations of a special sort in this theorem, namely horizontal ones. A foliation of a Seifert fibered manifold is horizontal if regular fibers are transverse to the leaves. In the picture above this would mean that $[h]$ is the class of a curve that cuts vertically through the plaques in M. *Co-orientable* means that the leaves of the foliation admit a coherent choice of normal vector. Here is an example of a standard construction used to produce foliations, for simplicity we consider a codimension one foliation of a surface.

Example 2.4. *If $f : S^1 \to S^1$ is an orientation-preserving homeomorphism, then construct a copy of the torus as*

$$T^2 \cong S^1 \times [0,1]/\sim$$

where $(x, 0) \sim (f(x), 1)$ for all $x \in S^1$. This torus naturally has a codimension one foliation whose leaves are in $1-1$ correspondence with orbits under f:

$$L = \bigcup_{n \in \mathbb{Z}} \{f^n(x)\} \times [0,1]$$

We will see that a construction similar to the one above is essential in understanding left-orderings of Seifert fibered manifolds.

Proof of Theorem 2.3. First we suppose that $\pi_1(M)$ is left-orderable, and we fix a corresponding dynamical realization $\hat{\rho} : \pi_1(M) \to \mathrm{Homeo}_+(\mathbb{R})$ [37]. Using the relators $\gamma_j^{\alpha_j} = h^{-\beta_j}$, one can argue that $\hat{\rho}(h)$ must act without fixed points. Thus $\hat{\rho}(h)$ is conjugate to one of $\rho(h)(x) = x + 1$ or $\rho(h)(x) = x - 1$. Without loss of generality, assume that $\hat{\rho}(h)$ conjugates to $\rho(h)(x) = x + 1$ and so $\hat{\rho}$ conjugates correspondingly to a representation $\rho : \pi_1(M) \to \mathrm{Homeo}_+(\mathbb{R})$.

As this point, observe that if M is constructed using the surface $\Sigma = \mathbb{R}P^2$, then there exists $y \in \pi_1(M)$ such that $yhy^{-1} = h^{-1}$, or in other words $hy = yh^{-1}$. But then we can calculate that for every x,

$$\underbrace{\rho(y)(x) < \rho(hy)(x)}_{\text{since } \rho(h)(x) = x+1} \quad = \quad \underbrace{\rho(yh^{-1})(x) < \rho(y)(x)}_{\text{since } \rho(h^{-1})(x) = x-1}$$

a contradiction. Thus $\Sigma = S^2$ and $h \in \pi_1(M)$ is central, therefore the image of ρ is contained within a certain subgroup of $\mathrm{Homeo}_+(\mathbb{R})$, namely

$$\rho : \pi_1(M) \to \widetilde{\mathrm{Homeo}_+}(S^1) = \{f \in \mathrm{Homeo}_+(\mathbb{R}) | f(x+1) = f(x) + 1\}$$

There is a second homomorphism of interest, which we will call ϕ

$$\phi : \pi_1(M) \to \pi_1(M)/\langle h \rangle$$

The group $\pi_1(M)/\langle h \rangle$ acts on a space X whose interior is homeomorphic to $\mathbb{R}^2$, specifically X is the "universal orbifold cover" of S^2 with singular points of order $\alpha_1 \dots, \alpha_n$

$$X \to S^2(\alpha_1, \dots, \alpha_n).$$

Construct a manifold $\widehat{M}$ as a quotient

$$\widehat{M} = X \times \mathbb{R} / \sim$$

where $\sim$ is defined by $(x, t) \sim (\phi(g)(x), \rho(g)(t))$. Then by construction, we get $\pi_1(M) \cong \pi_1(\widehat{M})$, and from this we can check that in fact M and $\widehat{M}$ are homeomorphic. So, this is a way of constructing our original manifold M, and this construction makes it clear that M admits a horizontal, co-orientable codimension one foliation:

The planes $X \times \{t\}$ descend to leaves as in the torus example, and the lines $\{x\} \times \mathbb{R}$ descend to Seifert fibers. The Seifert fibers are obviously transverse to the leaves, and provide a coherent choice of normal to the leaves as well.

Conversely, suppose that M admits a horizontal co-orientable foliation $\mathcal{F}$. Let $p : \widetilde{M} \to M$ be the universal cover, and $\widetilde{\mathcal{F}}$ the pullback foliation of $\widetilde{M}$. The fiber h in M pulls back to $p^{-1}(h) \cong \mathbb{R}$, and every leaf $\widetilde{L} \subset \widetilde{M}$ intersects $p^{-1}(h)$ transversely exactly once. Collapsing each leaf $\widetilde{L}$ to a point, we therefore get

$$\widetilde{M}/\widetilde{F} \cong \mathbb{R},$$

a copy of the reals (see [18] for full details). The action of $\pi_1(M)$ on $\widetilde{M}$ by deck transformations descends to an action on $\widetilde{M}/\widetilde{F}$, and "co-orientable" guarantees that the action will be order-preserving. Thus we have a representation

$$\rho : \pi_1(M) \to \mathrm{Homeo}_+(\mathbb{R})$$

and it follows that $\pi_1(M)$ is left-orderable by applying Theorem 1.4. □

Moreover, the works [18, 30, 36] and more recently [7] completely determine when the manifold $M(\pm g, b; \frac{\beta_1}{\alpha_1}, \dots, \frac{\beta_n}{\alpha_n})$ admits a co-orientable, horizontal foliation, and thus determines when $\pi_1(M)$ is left-orderable. Their condition is a system of diophantine inequalities involving the $\frac{\beta_i}{\alpha_i}$ which must have a solution if a foliation/left-ordering is to exist.

Thus we have proved that for Seifert fibered manifolds M, the fundamental group is left-orderable if and only if M admits a co-orientable taut foliation–but an inspection of the proof yields more. We have actually set up a correspondence between foliations of M and left-orderings of $\pi_1(M)$.

The same correspondence between orderings and foliations exists when the surface Σ used in the construction of M has boundary, as the construction of the proof can be modified to fit this case. Then if $\Sigma = D^2$, for instance, we get

$$\pi_1(M) = \langle \gamma_1, \ldots, \gamma_n, h | h \text{ central }, \gamma_j^{\alpha_j} = h^{-\beta_j} \rangle$$

By carefully keeping track of the boundary behaviour of the foliations, one can extend this correspondence between left-orderings and foliations to rational homology sphere graph manifolds W constructed by gluing together foliated Seifert fibered manifolds.

Theorem 2.5 (Boyer-C, [11]). *Let W be a rational homology 3-sphere graph manifold. Then $\pi_1(W)$ is left-orderable if and only if W admits a co-orientable taut foliation.*

This correspondence naturally raises a number of questions.

2.1. **Questions.** In the questions below, M is always Seifert fibered unless otherwise specified.

Q1. Several authors have shown that the groups

$$\pi_1(M) = \langle \gamma_1, \ldots, \gamma_n, h | h \text{ central }, \gamma_j^{\alpha_j} = h^{-\beta_j} \rangle$$

can admit isolated points in their space of left-orderings [17, 38, 28, 27], this is exactly the fundamental group you get when you start with $\Sigma = D^2$. How can the property of being 'isolated' be topologically characterized in terms of the corresponding foliation?

Remarks: As a specific example, the complement of the trefoil has fundamental group B_3, the braid group on three strands. The group B_3 admits an isolated ordering, namely the DD-ordering [17]. What can be said about the foliation which corresponds to the DD-ordering?

Q2. Is there an analogous correspondence for any other classes of 3-manifolds, perhaps via Thurston's universal circle construction for some class of hyperbolic manifolds?

Remarks: Suppose that M is an integer homology 3-sphere and M admits a co-orientable taut foliation. Then the universal circle gives us

$$\rho : \pi_1(M) \to \text{Homeo}_+(S^1)$$

and this homomorphism lifts to $\widetilde{\rho} : \pi_1(M) \to \widetilde{\text{Homeo}_+}(S^1)$ since $H^2(M)$ is trivial. This gives a way of constructing orderings from foliations, but is the correspondence reversible? Is there anything that can be done in the case that $H^2(M)$ is nontrivial, and so the representation may not lift? (See [6] for more information on the universal circle construction).

3. Manifolds with $|H_1(M)| < \infty$: Dehn filling and the L-space conjecture

What remains in our study is closed, connected, orientable, irreducible non-Seifert fibered manifolds with finite first homology. There is a conjecture which, if true, tells us exactly what to expect in these cases (indeed, the conjecture happens to hold in all the cases already covered).

Heegaard-Floer homology is a way of associating a finitely generated abelian group, $\widehat{HF}(M)$, to every closed, orientable, irreducible 3-manifold M. We will treat this homology theory as a black box, in the sense that we'll call upon theorems from this theory while giving no background on Heegaard-Floer homology itself. There are many introductions to the subject available, see [42]. An irreducible manifold M with $|H_1(M)| < \infty$ is called an L-space if

$$\mathrm{rank}\widehat{HF}(M) = |H_1(M)|.$$

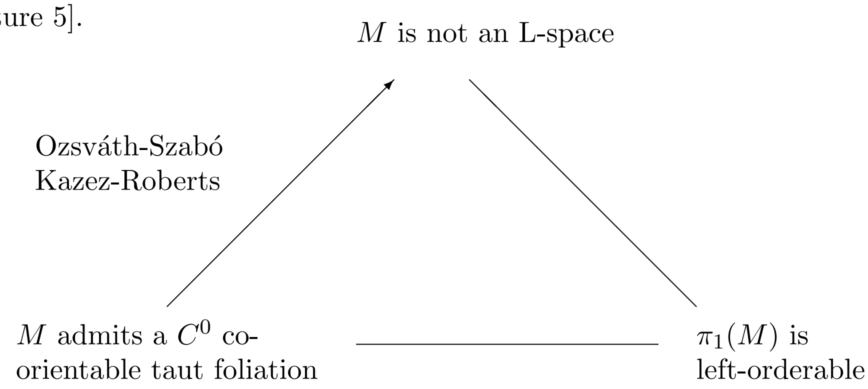
The following conjecture predicts which manifolds will have left-orderable fundamental group.

Conjecture 3.1 (Boyer-Gordon-Watson [2])**.** *An irreducible* 3*-manifold with* $|H_1(M)| < \infty$ *is an L-space if and only if its fundamental group is not left-orderable.*

In [2], the authors verify that the conjecture holds for all geometric non-hyperbolic manifolds, as well as some hyperbolic ones.

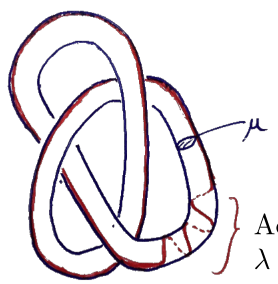
Example 3.2. *Lens spaces are L-spaces, as is any* 3*-manifold with finite fundamental group. In fact, since the conjecture has been verified for all Seifert fibered manifolds, every Seifert fibered M with $\pi_1(M)$ non-left-orderable is also an example of an L-space* [2]*.*

In general, this conjecture is part of a program to tie together orderings, foliations, and L-spaces. While we only understand the triangle below in some special cases, all sides are conjectured to be equivalences [31, Conjecture 5].



What we will do for this section is focus on a particular special case of interest, namely the 3-manifolds that arise from Dehn surgery on a knot in S^3. Our reason for choosing this class of manifolds is that Heegaard-Floer homology behaves particularly well with respect to Dehn surgery, so there are many theorems about L-spaces and Dehn surgery that should translate directly into something about left-orderability (if the conjecture is true).

First we recall the construction of Dehn surgery along a knot in S^3 (See [45]). Let $K \subset S^3$ be a knot, and remove a tubular neighbourhood $N(K)$ of K from S^3, think of $N(K) \cong S^1 \times D^2$. The curve $S^1 \times \{0\}$ is our original knot, and $\lambda \cong S^1 \times \{1\}$ is called a longitude (it is disjoint from and parallel to K). We choose λ so that $[\lambda] \in H_1(S^3 \setminus K)$ is the identity, this particular λ doesn't link the original knot K and is called the preferred longitude of K. The curve $\mu \cong \{1\} \times \partial D^2$ is called a meridian of K. See below for the standard longitude and a meridian of a trefoil, the longitude λ is in red.



The curves μ and λ represent classes in $\pi_1(\partial(S^3 \setminus N(K))) \cong \pi_1(T) \cong \mathbb{Z} \oplus \mathbb{Z}$ that generate the subgroup of the boundary torus.

For a fixed $[J] = \mu^p \lambda^q \in \pi_1(\partial N(K))$ we construct the manifold

$$S^3(K, \frac{p}{q}) = \left(S^3 \setminus \mathrm{int}(N(K))\right) \cup_\phi (S^1 \times D^2)$$

where $\phi : \partial(S^1 \times D^2) \to \partial N(K)$ sends $\{1\} \times \partial D^2$ to J. The fundamental group of the resulting manifold is

$$\pi_1(S^3(K, \frac{p}{q})) = \pi_1(S^3 \setminus K)/\langle\langle \mu^p \lambda^q \rangle\rangle$$

i.e., we set $\mu^p \lambda^q = 1$.

Mayer-Vietoris can be used to calculate that $H_1(S^3(K, \frac{p}{q})) \cong \mathbb{Z}/p\mathbb{Z}$, moreover these manifolds are almost always irreducible (this can be made very precise). So, we have a way of building many manifolds with $|H_1(M)| < \infty$ in order to test the conjecture.

Recall that Heegaard-Floer homology is very "well-behaved" with respect to Dehn surgery. To show what is meant by this, below is a sample of a few of the available theorems concerning L-spaces and Dehn surgery, with corresponding 'translations' into conjectures about left-orderability.

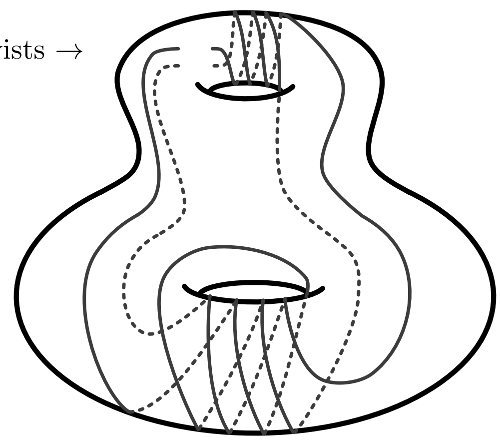
3.1. **Non-left-orderable fillings behave like L-spaces.**

Theorem 3.3 (Ozsváth-Szabó, consequence of triad condition, [41])**.** *Let $K \subset S^3$, and suppose that $S^3(K, \frac{p}{q})$ is an L-space. Then $S^3(K, \frac{p'}{q'})$ is also an L-space whenever $\frac{p'}{q'} > \frac{p}{q}$.*

So what *should* be true if we believe the conjecture is:

Conjecture 3.4. *Let $K \subset S^3$, and suppose that $\pi_1(S^3(K, \frac{p}{q}))$ is not left-orderable. Then $\pi_1(S^3(K, \frac{p'}{q'}))$ is also not left-orderable whenever $\frac{p'}{q'} > \frac{p}{q}$.*

Discussion. This behaviour has been verified for several infinite families of knots [14, 15], in particular torus knots, some cable knots and the knots $T^m_{3,5}$, which appear as in the figure below.



It is known that the knots $T^m_{3,5}$ satisfy the following property: There exists $r \in \mathbb{Q}$ such that for all $\frac{p}{q} \in [r, \infty]$, the manifold $S^3(K, \frac{p}{q})$ is an L-space. Correspondingly there should be $r \in \mathbb{Q}$ such that the group $\pi_1(S^3(K, \frac{p}{q}))$ is non-left-orderable whenever $\frac{p}{q} \in [r, \infty]$. In [15], it is shown that indeed such an r exists, specifically for $\frac{p}{q} \geq 15 + 2m$, the result of $\frac{p}{q}$-surgery on the knot $T^m_{3,5}$ is a manifold with non-left-orderable fundamental group.

At present I do not know of any published strategies that work towards proving this conjecture, only of strategies to verify that the necessary property holds for a given knot. Perhaps it would be easier to prove the weaker conjecture:

Conjecture 3.5. *Let $K \subset S^3$, and suppose that $\pi_1(S^3(K, \frac{p}{q}))$ is* ***finite****. Then $\pi_1(S^3(K, \frac{p'}{q'}))$ is also not left-orderable whenever $\frac{p'}{q'} > \frac{p}{q}$.*

3.2. **Non-left-orderable fillings detect knot genus.** For a fixed knot $K \subset S^3$, Heegaard-Floer homology also gives the exact value of the smallest slope $\frac{p}{q} \in \mathbb{Q}$ such that $S^3(K, \frac{p}{q})$ is an L-space. Let $g(K)$ denote the smallest genus of a surface $\Sigma \subset S^3$ such that $\partial\Sigma = K$. Then $g(K)$ is the knot genus, and we have a theorem:

Theorem 3.6 (Ozsváth-Szabó, [41, 40])**.** *Let $K \subset S^3$, and suppose that there exists $\frac{p}{q} \in \mathbb{Q}$ such that $S^3(K, \frac{p}{q})$ is an L-space. Then $S^3(K, \frac{p'}{q'})$ is an L-space if and only if $\frac{p'}{q'} > 2g(K) - 1$.*

Correspondingly, we should have

Conjecture 3.7. *Let $K \subset S^3$, and suppose that there exists $\frac{p}{q} \in \mathbb{Q}$ such that $\pi_1(S^3(K, \frac{p}{q}))$ is not left-orderable. Then $\pi_1(S^3(K, \frac{p'}{q'}))$ is not left-orderable if and only if $\frac{p'}{q'} > 2g(K) - 1$.*

Discussion. For all known examples of knots with non-left-orderable Dehn fillings, aside from torus knots, there is a slight 'gap'. For example, the knot $T^1_{3,5}$ (using the notation of the previous section) has $g(T^1_{3,5}) = 5$ (it is the $(-2, 3, 7)$ pretzel knot, or $12n_{0242}$ in DT-notation), and so the conjecture predicts:

- Slopes that should yield left-orderable groups: $\frac{p}{q} \in (-\infty, 9)$
- Slopes that should yield non-left-orderable groups: $\frac{p}{q} \in [9, \infty]$

However the computation in [15] only gave non-left-orderability for $\frac{p}{q} \in (17, \infty]$. Implicit in the work of [32] is an improvement which shows non-left-orderability for $\frac{p}{q} \in (10, \infty]$. So there is a gap $\frac{p}{q} \in [9, 10]$ where non-left-orderability is unknown, though from the conjecture above we know exactly what is predicted (Of course we're also missing left-orderability of *all* of $(-\infty, 9)$).

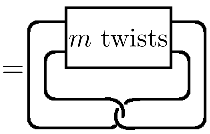
3.3. **Alternating knots always give left-orderable Dehn fillings.** A knot K is called alternating if, for some oriented diagram of K, you alternately encounter over and under crossings as you travel around the knot (following the orientation of the diagram). A knot K is called hyperbolic if $S^3 \setminus K$ a complete Riemannian metric of constant negative curvature, i.e. it has hyperbolic geometry.

Theorem 3.8 (Ozsváth-Szabó, [41])**.** *If K is an alternating hyperbolic knot, then no manifold $S^3(K, \frac{p}{q})$ is an L-space.*

Conjecture 3.9. *If K is an alternating hyperbolic knot, then every group $\pi_1(S^3(K, \frac{p}{q}))$ is left-orderable.*

Discussion. Most left-orderability and Dehn surgery results could be considered as working towards this conjecture, since many of the computed examples deal with alternating hyperbolic knots. Here are some of the known results:

- The figure eight knot has left-orderable surgery quotients whenever $\frac{p}{q} \in [-4, 4] \cup \mathbb{Z}$, and the figure eight knot is a hyperbolic alternating knot [2, 19].
- Consider the twist knots

$K_m =$ 

where $m \geq 2$. These knots are alternating and hyperbolic, and whenever $\frac{p}{q} \in [0, 4]$, the manifold $S^3(K_m, \frac{p}{q})$ has a left-orderable fundamental group [23].

- More generally, two-bridge knots are alternating and many (most) of them are hyperbolic. Every two-bridge knot has an interval of left-orderable slopes that contains zero [47].
- Teragaito and Motegi also construct infinite families of alternating knots for which every surgery yields a left-orderable fundamental group [35] (Note that the article [35] also has a good introduction that covers the current state of Dehn filling and its relationship with left-orderability).
- A few other examples [22].

In all of these cases, the strategy to produce left-orderable fillings is to prove that there exists a continuously varying one-parameter family of representations $\rho_t : \pi_1(S^3 \setminus K) \to \mathrm{SL}(2, \mathbb{R})$ and an interval of slopes I such that for every $\frac{p}{q} \in I$, there exists $t_0 \in I$ such that

- $\rho_{t_0}(\mu^p \lambda^q) = 0$, and
- The ρ_t can be lifted to representations $\tilde{\rho}_t : \pi_1(S^3 \setminus L) \to \widetilde{\mathrm{SL}(2, \mathbb{R})}$ which satisfy $\tilde{\rho_{t_0}}(\mu^p \lambda^q) = 0$.

From this it follows that $\frac{p}{q}$ surgery gives a left-orderable group, by applying Theorem 1.4 and using the fact that $\widetilde{\mathrm{SL}(2, \mathbb{R})}$ is left-orderable.

3.4. Fiberedness, bi-orderability and left-orderable quotients.

Theorem 3.10. *If K admits a slope $\frac{p}{q}$ such that $S^3(K, \frac{p}{q})$ is an L-space, then*

(1) *K is fibered* [20, 39]

(2) *The Alexander polynomial of K has the form*

$$\Delta_K(t) = (-1)^k + \sum_{j=1}^{k} (-1)^{k-j}(t^{n_j} + t^{-n_j})$$

for some increasing sequence of integers $0 < n_1 < n_2 < \ldots < n_k$ [41].

Obviously here we can replace L-space by non-left-orderable and get another conjecture, the most notable being that the existence of a non-left-orderable Dehn filling of K should imply that K is fibered. However I would like to point out another consequence, which relates to the bi-orderability material from the first section.

Corollary 3.11 (C-Rolfsen [13]). *If $\pi_1(S^3 \setminus K)$ is bi-orderable, then no manifold $S^3(K, \frac{p}{q})$ is an L-space.*

Proof. Suppose $\pi_1(S^3 \setminus K)$ is bi-orderable and that $S^3(K, \frac{p}{q})$ is an L-space for some $\frac{p}{q} \in \mathbb{Q}$. Then by (1) above, K is a fibered knot. Then Theorem 1.10(2) applies, and we conclude that $\Delta_K(t)$ must have a positive real root since $\pi_1(S^3 \setminus K)$ is bi-orderable. This is a contradiction, because no polynomial of the form given in (2) above can have a positive real root [13]. □

Replacing L-space with non-left-orderable in the above theorem we get

Conjecture 3.12. *If $\pi_1(S^3 \setminus K)$ is bi-orderable, then for every $\frac{p}{q} \in \mathbb{Q}$ the group $\pi_1(S^3(K, \frac{p}{q}))$ is left-orderable.*

Discussion. There are no results that I know of specifically addressing this conjecture, though the twist knots with an even number of twists all have bi-orderable knot groups and also an interval of left-orderable slopes ($[0, 4]$ for every knot except the figure eight knot, where it's $[-4, 4]$, as we saw above).

This leads naturally to the question

Question 3.13. *Suppose that G is a bi-orderable group and let $1 \neq q \in G$ be given. If the group $G/\langle\langle g \rangle\rangle$ is torsion free, is it left-orderable?*

If the answer is "yes" then the conjecture above has nothing to do with low dimensional topology, and is a purely group-theoretic fact. On the other hand, if the answer is "no" then an example of a bi-orderable G with torsion-free non-left-orderable quotient $G/\langle\langle g \rangle\rangle$ would give insight into what properties unique to 3-manifold groups may be needed to prove the conjecture above (if it is true).

Update: This question was answered by Ying Hu, who gives the following construction of a torsion-free non-left-orderable quotient: Start with the figure eight knot group, which is bi-orderable. The fundamental group G of the 3-fold cyclic cover of this knot has a bi-orderable fundamental group, since it is a subgroup of index 3 in a bi-orderable group. Now take the quotient of G by the third power of the knot meridian to obtain the fundamental group of the cyclic *branched* cover of the figure eight. This is a torsion-free group, but recent work of Gordon and Lidman [21, Corollary 1.11] shows that this group is not left-orderable.

Note that this question is related to the theorem of Brodskii and Howie:

Theorem 3.14 (Brodskii [4], Howie [25]). *Torsion-free one relator groups are locally indicable.*

The question above is related in that we have replaced "free groups" with "bi-orderable groups" and "locally indicable" with "left-orderable".

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