A RESEARCH CHALLENGE

Abstract. Here is a (probably not too difficult) calculational problem whose solution would be very interesting. If you want to learn more about why a solution would be interesting (it's not obvious why), or if you have a solution please contact `Adam.Clay@umanitoba.ca`. Given proper context, a solution to this problem could potentially be published!

A group G is left-orderable if there exists a strict total ordering $<$ of the elements of G, such that $g < h$ implies $fg < fh$ for all $f, g, h \in G$. For example, groups like $\mathbb{Z}, \mathbb{Q}$ and $\mathbb{R}$ with addition are all obviously left-orderable.

Writing 1 for the identity in G, an element g satisfying $1 < g$ is called positive, if $g < 1$ is it called negative. If G is left orderable and $1 < g$ then this forces $g^{-1} \cdot 1 < g^{-1} \cdot g$, or simply $g^{-1} < 1$. In other words, if an element is positive then its inverse is negative, and vice versa.

Showing that a group is left-orderable is generally hard. Showing that a group is not left-orderable is generally easier. The approach to proving non-left-orderability is always to assume that there is a left-ordering, and then arrive at a contradiction. Here is an example of this.

Proposition 0.1. *If a group G has torsion (i.e. there exists a non-identity element $g \in G$ such that $g^k = 1$ for some $k > 0$), then G is not left-orderable.*

Proof. Suppose that G is left-orderable and that there is some $g \in G$ with $g \neq 1$ but $g^k = 1$. There are two cases, either $1 < g$ or $g < 1$, which we treat separately.
Case 1. $1 < g$. Then since left-multiplication preserves the ordering, by left multiplying by g we get $g < g^2$, left multiplying again we get $g^2 < g^3$, etc... Stringing together all the inequalities we get in this way, we arrive at

$$1 < g < g^2 < g^3 < \cdots < g^k = 1$$

and we get $1 < 1$, a contradiction.
Case 2. $g < 1$. In this case we left multiply by g we get $g^2 < g$, $g^3 < g^2$, etc... and then

$$1 = g^k < \cdots < g^3 < g^2 < g < 1$$

again a contradiction. □

All of our arguments of this type can be simplified by observing that we can 'reverse' a left-ordering of a group G to make a new left-ordering. Explicitly, if G is a group and $<$ is an ordering of its elements, then we can define a new ordering $<'$ by

$$g <' h \text{ iff } h < g.$$

So in the proof above, we don't need to do two cases. If you choose an element $g \in G$ and you assume there exists a left-ordering of G, you can also assume that $1 < g$, because if $g < 1$, you can just reverse the ordering to get $1 < g$.

Here is a more complicated example of a group that is not left-orderable. Before we begin, we introduce an obvious lemma: The product of positive elements is positive, because if $1 < g$ and $1 < h$ then $h < hg$ by left multiplication, and by combining inequalities $1 < h < hg$.

Example 0.2. The group with presentation

$$G = \langle a, b : bababa^{-1}b^2a^{-1} = 1, \quad ababab^{-1}a^2b^{-1} = 1\rangle$$

is not left-orderable.

To see that G is not left-orderable, assume that it is left-orderable with ordering $<$. By reversing the ordering if necessary, we may assume that a is positive. We make a case argument as follows.

Case 1: b is positive. Then all combinations of a and b with positive exponents are positive.

Subcase 1.1: ab^{-1} is positive. Then $1 < abab \cdot ab^{-1} \cdot a \cdot ab^{-1}$ because it is a product of positive elements. But one of the defining equations of the group is $ababab^{-1}a^2b^{-1} = 1$. So in this subcase we have $1 < 1$, a contradiction.

Subcase 1.2: ba^{-1} is positive. Then $1 < baba \cdot ba^{-1} \cdot b \cdot ba^{-1}$. Again, one of the defining relations of G is $bababa^{-1}b^2a^{-1} = 1$, so we get $1 < 1$.

Case 2: b is negative. To simplify notation, write $R_1 = bababa^{-1}b^2a^{-1}$ and $R_2 = ababab^{-1}a^2b^{-1}$. Then since $R_1 = 1$ and $R_2 = 1$, we calculate $1 = b^{-1}R_1^{-1}bR_2 = b^{-1}ab^{-2}a^2b^{-1}a^2b^{-1}$. But this latter word, being a product of a and b^{-1}, must be positive. Again $1 < 1$, a contradiction.

Thus in any case we are led to a contradiction by assuming left-orderability. □

Obviously this case-by case argument involves several clever choices: Choosing to ab^{-1} positive and ab^{-1} negative as the subcases of Case 1, or thinking to multiply together $b^{-1}R_1^{-1}bR_2$ in Case 2 are both clever choices that made the proof quite short.

Can you find a clever proof of the following fact:

Theorem 0.3. *Suppose that p and q positive, relatively prime integers. Whenever $p/q > 9$, the following group is not left-orderable:*

$$\langle a, b : a^2b^{-1}a^2b^{-2}a^{-1}b^{-2} = 1, (b^2a^{-1})^p(b^2aba(b^2a^{-1})^{-17})^q = 1\rangle$$

This has been solved when $p/q \geq 17$, in Section 4.2 of the paper "Left-orderable fundamental groups and Dehn surgery" by Clay and Watson. So really, the question is interesting for $9 \leq p/q < 17$. Even if you can only get it to work for $p = 9, q = 1$ (i.e. $p/q = 9$) that would still be great, because smaller fractions p/q are probably going to be more difficult for some reason.