Recall from the previous talk by Derek that <u>symplectic geometry</u> is a branch of differential Geometry originated as the mathematical tool of classical mechanics. It is a branch of differential geometry studying symplectic manifolds.

Example: The cotangent bundle T^*M of

a manifold M, endowed with the form $\omega = d\theta$, where θ is the 1-form the "tautological" 1-form. That is,

$\theta(x,p) = \pi^* p$, $\forall (x,p) \in T^*M$, with

$$\begin{aligned} \pi : T^*M &\longrightarrow M \\ (x,p) &\longmapsto x \end{aligned}$$

$$\theta(x,p) = \pi^* p \quad \forall (x,p) \in T^*M,$$

$$\Leftrightarrow \ \iota_{v_p} \theta = p(\pi_* v) \quad \Rightarrow$$

In local coordinates (x^i, ξ_i),

$$\theta = \sum_{i=1}^{n} \xi_i \, dx^i$$

In classical mechanics, symplectic geometry provides a robust mathematical structure for understanding the dynamics of point particles. However, classical field theories, which describe continuous fields rather than discrete particles, require a more sophisticated geometrical framework.

Based on the previous talk, we can have the following dictionary between classical mechanics and symplectic geometry.

In classical mechanics, the state of a system is described by its position and momentum. This is mathematically represented by the phase space, which is a symplectic manifold in symplectic geometry. Each point on this manifold corresponds to a possible state of the mechanical system.

A symplectic manifold (M, ω) comes equipped with a closed, nondegenerate 2-form ω, known as the symplectic structure. This form encapsulates the conservation of volume in phase space and is fundamental in defining the dynamics of the system.

The dynamics of a mechanical system are often described by a Hamiltonian function, which represents the total energy (kinetic + potential) of the system. In symplectic geometry, this function generates a flow on the manifold that time evolution of the system.

In symplectic geometry, trajectories of particles are represented as curves $\gamma : \mathbb{R} \longrightarrow M$.

In classical mechanics, the momentum at a point is a covector in T^*M.

In mechanics, canonical transformations preserve the structure of Hamilton's equation and are central to simplifying problems. In symplectic geometry, these are transformations that preserve the symplectic form, known as symplectomorphism.

Noether's theorem relates symmetries of the physical system to conservation laws. In symplectic geometry, symmetries correspond to conserved quantities (like energy, momentum) and are associated with invariance properties

• Configuration space	• smooth manifold M
• Trajectories	• smooth curves $\gamma : \mathbb{R} \to M$
• velocities	• Tangent vector $v \in TM$
• Inertia	• Riemannian structure over M
• Free motions	• Geodesics wrt g
• Mech momentum	• $p \in T^*M$
• Phase space	• T^*M
• Measurable quantities (observables)	• $f \in C^\infty(T^*M)$
• conservation of volume	→ symplectic structure
Total energy: kinetic + potential	• Hamiltonian function

• Solutions of Hamiltonian equations	• Integral curves of Hamiltonian vector fields
• Time evolution of observables	⇝ Poisson brackets
• canonical transformat°	• symplectomorphisms
• conserved quantities (energy, momentum)	• symmetries

Let (M, ω) be a symplectic manifold.
The symplectic structure provides a criterion for selecting relevant classes of smooth objects :

- Hamiltonian vector fields
- Hamiltonian forms
- Group action (moment map)

<u>Def</u>: Let $f \in C^\infty(M)$. A Hamiltonian vector associated with f is a vector field $v_f \in \mathfrak{X}(M)$ such that

$$\iota_{v_f} \omega = -df \qquad (\text{i.e. } \iota_{v_f} \omega \text{ is exact})$$

Note: Hamiltonian vector fields represent the infinitesimal generators of the time evolution of the system, encapsulating how the state of the system changes over time, The Hamiltonian typically represents the total energy: kinetic + potential. The Hamiltonian vector field thus describes how the system evolves in a way that conserves this total energy.

Energy: $f \longmapsto v_f \longmapsto$ —

Time evolution generated by f.

Dually to the geometric approach, there is an algebraic approach: Lie algebra of observables.

- observables ⇝ smooth functions $C^\infty(M)$
- Poisson bracket: The key algebraic structure is given by the Poisson bracket

This operation is bilinear, skew symmetric and satisfies the Jacobi identity.

making $(C^\infty(M), \{\cdot,\cdot\})$ into a Lie algebra.

Multisymplectic or n-plectic structures are a generalization of symplectic structures where the fixed form is allowed to have degree > 2.

An n-plectic manifold (M, ω) is a manifold M endowed with a closed, non-degenerate $(n+1)$-form ω.

a) For $n = 1$, (M, ω) is a symplectic manifold. So 1-plectic manifold is just a symplectic. Hence n-plectic structures are a generalization of symplectic structures.

b) An $(n+1)$-dimensional orientable manifold equipped with a volume form is an n-plectic manifold.

c) Any compact semi simple Lie group G is a 2-plectic manifold when equipped with the canonical bi-invariant 3-form (Cartan 3-form)

$$\nu(x, y, z) = \langle x, [y, z] \rangle, \quad \forall x, y, z \in \mathfrak{g}$$

where $\langle\ ,\ \rangle$ is the Killing form, and $[\ ,\ ]$ is the natural bracket on $\mathfrak{g}$.

d) Let (M, g) be a Riemannian manifold which admits two anticommuting, almost complex structure $J_1, J_2 : TM \to TM$ i.e., $J_1^2 = J_2^2 = -\mathrm{Id}$ and $J_1 J_2 = -J_2 J_1$.

Then $J_3 = J_1 J_2$ is also an almost complex structure.

If J_1, J_2, J_3 preserve the metric g, then one can define the 2-forms $\theta_1, \theta_2, \theta_3$, where

If θ_i is closed, then M is called a hyper Kähler manifold.

Given such a manifold, one can construct the 4-form

One can show that ω is closed and nondegenerate.

Hence hyper Kähler manifold is a 3-plectic manifold.

Let Q be a manifold. Then the multi-cotangent bundle $M = \Lambda^n T^* Q$ is an n-plectic manifold with $\omega = d\theta$ where

$$\theta_\alpha (x_1, \dots, x_n) = \alpha (\pi_* x_1, \dots, \pi_* x_n)$$

$$TM \longrightarrow M$$

Multisymplectic geometry originated from the classical field theory, just as symplectic geometry originated from classical mechanics.

More precisely, given $(n+1)$-dim'l classical fied, one can construct a finite dim'l $(n+1)$-plectic manifold known as the "multiphase space".

The relationship between multisymplectic geometry and classical field theories can be understood through several key aspects:

In classical field theories, the concept of phase space is generalized to a multisymplec manifold.

capture the relationship between field variables and their derivatives.

Just as symplectic geometry naturally leads to Hamilton's equations in classical mechanics, multisym-

plectic geometry gives rise to the field equations in field theories.
These equations describe how field configurations evolve over time and space

Multisymplectic geometry elegantly encapsulates the conservation laws and symmetries inherent in field theories. These conservation laws are reflected in the properties of the multisymplectic form

Unlike symplectic manifolds, which are inherently even-dimensional due to the non-deg, closed 2-form defining their structure, n-plectic manifolds are not bound by this even dimensionality constraint.

This flexibility flexibility in dimensionality allows symplectic manifolds to model a wider variety of physical systems, but also leads to complexity in their matematical structure.

one of the cornerstone of symplectic geometry is the Darboux theorem, which states that around any point on a symplectic manifold, there exist local coordinates (Darboux coordinates) in which the symplectic form takes a standard form. This theorem implies a strong local homogeneity for symplectic manifolds.

In contrast, n-plectic manifolds generally lack an analogue of Darboux coordinates, meaning they lack a standardized local form. This leads to a richer and more varied local geometry compared to the uniform local structure of symplectic manifolds.

The absence of a Darboux like

theorem results in a multitude of local models for n-plectic manifolds, contrasting with local isomorphism property of symplectic manifolds. This diversity makes n-plectic manifolds inherently more complex and variable in their local properties.

Let (M,ω) be an n-plectic manifold. An $(n-1)$-form α is iff there exists a vector field $v_\alpha \in \mathfrak{X}(M)$ such that

$$d\alpha = -i_{v_\alpha}\omega .$$

- v_α = "Hamiltonian vector corresponding to α"
- $\Omega^{n-1}_{Ham}(M,\omega) = \{ \alpha \in \Omega^{n-1} \mid \exists\, v_\alpha \in \mathfrak{X}(M) : d\alpha = -i_{v_\alpha}\omega \}$
- $\mathfrak{X}_{Ham}(M) = \{ v \in \mathfrak{X}(M) \mid \exists\, \alpha \in \Omega^{n-1} : d\alpha = -i_{v_\alpha}\omega \}$
- Note: $\Omega^{n-1}_{Ham}(M,\omega)$ and $\mathfrak{X}_{Ham}(M)$ are both vector spaces.

If v_α is a Hamiltonian v.f, then
$$\mathcal{L}_{v_\alpha}\omega = 0$$

<u>Proof</u>:

$$\mathcal{L}_{v_\alpha}\omega = (\iota_{v_\alpha} d + d\,\iota_{v_\alpha})\,\omega = \iota_{v_\alpha}\underbrace{d\omega}_{0} + d\,\iota_{v_\alpha}\omega$$
$$= d(-d\alpha) = -d^2\alpha = 0$$

Now we define a bracket on $\Omega^{n-1}_{Ham}(M)$ that generalizes the Poisson bracket.

The map
$$\{\cdot,\cdot\} : \Omega^{n-1}_{Ham}(M) \times \Omega^{n-1}_{Ham}(M) \longrightarrow \Omega^{n-1}_{Ham}(M)$$
$$(\alpha,\beta) \longmapsto \{\alpha,\beta\} = \iota_{v_\beta}\iota_{v_\alpha}\omega$$
defines a bracket.

- When $n=1$, the bracket is the usual Poisson bracket of smooth functions.
- For $n>1$, this bracket does not need to satisfy the Jacobi identity.

Let $\alpha, \beta \in \Omega^{n-1}_{Ham}(M)$, v_α, v_β be their respective h.v.f. Then

(1) $\{\alpha, \beta\} = -\{\beta, \alpha\}$ (skew symmetry)

(2) The bracket of Hamiltonian forms is Hamiltonian:

$$d\{\alpha, \beta\} = -\iota_{[v_\alpha, v_\beta]}\omega$$

$$\Rightarrow v_{\{\alpha, \beta\}} = [v_\alpha, v_\beta]$$

<u>Proof:</u> 1) ok

2) $d\{\alpha, \beta\} = \underbrace{d\, i_{v_\beta}}\, i_{v_\alpha}\, \omega$

$$= (\mathcal{L}_{v_\beta} - i_{v_\beta} d)\, i_{v_\alpha}\omega$$

$$= \mathcal{L}_{v_\beta} i_{v_\alpha}\omega - i_{v_\beta} d\, i_{v_\alpha}\omega$$

$$= \mathcal{L}_{v_\beta} i_{v_\alpha}\omega - i_{v_\beta} d\,(\underbrace{-d\alpha}_{0})$$

$$= \mathcal{L}_{v_\beta} i_{v_\alpha}\omega$$

$$= i_{[v_\alpha, v_\beta]}\omega + i_{v_\alpha}\underbrace{\mathcal{L}_{v_\beta}\omega}_{0}$$

$$= i_{[v_\alpha, v_\beta]}\omega$$

The bracket $\{\cdot,\cdot\}$ satisfies the bracket up to an exact $(n-1)$-form:

$$\{\alpha_1, \{\alpha_2, \alpha_3\}\} + \{\alpha_2, \{\alpha_3, \alpha_1\}\} + \{\alpha_3, \{\alpha_1, \alpha_2\}\}$$
$$= - d\,\iota(v_{\alpha_1} \wedge v_{\alpha_2} \wedge v_{\alpha_3})\,\omega$$

In n-plectic geometry the nature of the bracket operation exhibit unique characteristic that differ significantly from the familiar structures in symplectic geometry. Key aspects of this bracket in n-plectic geometry include:

unlike the Lie bracket found in symplectic geometry, the bracket in n-plectic geometry does not satisfy the Jacobi identity.

In $(C^\infty(M), \{\cdot,\cdot\})$, we can multiply functions by point-wise multiplication:

$$(fg)(x) = f(x)\, g(x)$$

We lose this property in n-plectic geometry.

Interestingly, while the n-plectic bracket does not satisfy the Jacobi identity, it descends to an honest Lie bracket on $\Omega^{n-1}_{Ham}(M) / d\,\Omega^{n-2}(M)$

An additional and notable feature of n-plectic geometry is the existence of brackets with arities different than 2. This means the bracket operation can take more than 2 inputs, a stark contrast to the binary operations in symplectic geometry.

is a graded vector space

L equipped with a collection of map

$\{ l_k : L^{\otimes k} \longrightarrow L \mid 1 \leq k < \infty \}$

of skew symmetric maps with $|l_k| = k-2$

s.t the following identity holds for

$1 \leq m < \infty$

$$\sum_{\substack{i+j=m+1 \\ \sigma \in Sh(i,m-i)}} (-1)^{\sigma} \varepsilon(\sigma) (-1)^{i(j-1)} l_j (l_i (x_{\sigma(1)}, \dots, x_{\sigma(i)}), x_{\sigma(i+1)}, \dots, x_{\sigma(m)}) = 0$$

An L_∞-algebra (L, ℓ_k) is a

iff the underlying graded vector space L is concentrated in degrees $0, 1, \dots, n-1$.

It is a chain complex

consider n-plec (M, ω)

$\overset{n}{C^\infty}(M)$

$[\cdot, \dots, \cdot]_k$

with n (skew symmetric) multibrackets

$(2 \le k \le n+1)$

$$[\cdot, \dots, \cdot]_k : \left(\Omega^{n-1}_{Ham}\right)^k \longrightarrow \Omega^{n+1-k}$$

$$\alpha_1 \otimes \cdots \otimes \alpha_n \longmapsto (-1)^{k+1}\, \iota_{v_{\alpha_1}} \cdots \iota_{v_{\alpha_k}} \omega$$

A Lie 2-algebra is defined by the data of the following

(a) A 2-term chain complex of vector space $L_\bullet = (L_1 \xrightarrow{d} L_0)$

where d is the differential

(b) A chain map $[\cdot,\cdot] : L \otimes L \longrightarrow L$ called the brackets

(c) A chain homotopy $S : L \otimes L \longrightarrow L$ from the chain map $L \otimes L \longrightarrow L$

$x \otimes y \longmapsto [x,y]$

to the chain map $L \otimes L \longrightarrow L$

$x \otimes y \longmapsto -[y,x]$

(c) A skewsymmetric chain homotopy

$J : L_\bullet \otimes L_\bullet \otimes L_\bullet \longrightarrow L_\bullet$

which serves as a transformation between two chain maps. specifically, it maps

$$L_\bullet \otimes L_\bullet \otimes L_\bullet \longrightarrow L_\bullet$$
$$x \otimes y \otimes z \longmapsto [x, [y, z]]$$

to

$$L_\bullet \otimes L_\bullet \otimes L_\bullet \longrightarrow L_\bullet$$
$$x \otimes y \otimes z \longmapsto [[x, y], z] + [y, [x, z]]$$

(Jacobiator)

d) The Jacobiator must fulfill a compatibility condition given by

$$[x, J(x,y,w)] + J(x,[y,z],w) + J(x,z,[y,w])$$
$$+ [J(x,y,z),w] + [z, J(x,y,w)]$$
$$= J(x,y,[z,w]) + J([x,y],z,w) +$$
$$[y, J(x,z,w)] + J(y,[x,z],w)$$
$$+ \quad J(y,z,[x,w])$$

Let (M, ω) be a 2-plectic manifold
Then there exists a Lie 2-algebra
$L(M, \omega)$ given by

- $L_0 = \Omega^1_{Ham}(M)$
- $L_1 = C^\infty(M)$
- $d : C^\infty(M) \longrightarrow \Omega^1_{Ham}$ is the exterior diff.
- the alternator $S : \Omega^1_{Ham} \otimes \Omega^1_{Ham} \longrightarrow C^\infty$
$(\alpha, \beta) \longmapsto 0$

- The Jacobiator is the trilinear map

$$J : \Omega^1_{Ham} \otimes \Omega^1_{Ham} \otimes \Omega^1_{Ham} \longrightarrow C^\infty(M)$$

$$(\alpha, \beta, \gamma) \longmapsto -\iota_{v_\alpha} \iota_{v_\beta} \iota_{v_\gamma} \omega$$