12.2 A sufficient condition for differentiability.

Continuity of all but one of the partials implies differentiability (together with existence in an open ball).

Theorem 12.11: Assume that $f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ and that $D_1 f, \ldots, D_n f$ exist at $\vec{c} \in \mathbb{R}^n$, and that $n-1$ of them additionally exist in some ball $B(\vec{c})$ and are continuous at $\vec{c}$. Then f is differentiable at $\vec{c}$.

Proof: First note: a vector-valued function $f = (f_1, \ldots, f_m)$ is differentiable at $\vec{c} \in \mathbb{R}^n$ iff each component f_i is differentiable at $\vec{c} \in \mathbb{R}^n$. This follows from considering the Taylor formula

$$f(\vec{c}+\vec{v}) - f(\vec{c}) = f'(\vec{c})(\vec{v}) + \|\vec{v}\| E_c(\vec{v})$$

component-wise, and observing that $\lim_{\vec{v} \to 0} E_c(\vec{v}) = 0$ if and only if all components of $E_c(\vec{v})$ go to zero as $\vec{v} \to 0$.

So we will prove the theorem under the assumption $f: \mathbb{R}^n \longrightarrow \mathbb{R}$. For ease of exposition, we'll assume that $D_1 f(\vec{c})$ exists and that the partials $D_2 f, \ldots, D_n f$ exist in some ball $B(\vec{c})$ and are continuous at $\vec{c}$.

We'll prove

$$f(\vec{c}+\vec{v}) - f(\vec{c}) = \nabla f(\vec{c}) \cdot \vec{v} + \|\vec{v}\| E(\vec{v}) \text{ with}$$

$$\lim_{\vec{v} \to 0} E(\vec{v}) = \vec{0}.$$

Rewrite $\vec{v} = \lambda \vec{y}$, where $\|\vec{y}\| = 1$ and λ is small enough that $\vec{c}+\vec{v} \in B(\vec{c})$. Write

$$\vec{y} = y_1\vec{u}_1 + y_2\vec{u}_2 + \dots + y_n\vec{u}_n \quad (\vec{u}_i\text{'s unit vectors}).$$

Now

$$
\begin{aligned}
f(\vec{c}+\vec{v}) - f(\vec{c}) &= f(\vec{c} + \lambda \vec{y}) - f(\vec{c}) \\
&= f(\vec{c} + \lambda(y_1\vec{u}_1)) - f(\vec{c} + \vec{0}) \\
&\quad + f(\vec{c} + \lambda(y_1\vec{u}_1 + y_2\vec{u}_2)) - f(\vec{c} + \lambda(y_1\vec{u}_1)) \\
&\quad + f(\vec{c} + \lambda(y_1\vec{u}_1 + y_2\vec{u}_2 + y_3\vec{u}_3)) - f(\vec{c} + \lambda(y_1\vec{u}_1 + y_2\vec{u}_2)) \\
&\quad + \\
&\quad \vdots \\
&\quad + f\Big(\vec{c} + \lambda\sum_{i=1}^{n} y_i\vec{u}_i\Big) - f\Big(\vec{c} + \lambda\sum_{i=1}^{n-1} y_i\vec{u}_i\Big).
\end{aligned}
$$

Consider the first term $f(\vec{c} + \lambda y_1\vec{u}_1) - f(\vec{c})$. The points $\vec{c} + \lambda y_1\vec{u}$ and $\vec{c}$ differ in their first coordinate only. Thus

$$f(\vec{c} + \lambda y_1\vec{u}_1) - f(\vec{c}) = \lambda y_1 D_1 f(\vec{c}) + \lambda y_1 E_1(\lambda)$$

with $\lim_{\lambda \to 0} E_1(\lambda) = 0$.

Considering the k^{th} term in the sum, $k \geq 2$, we have

$f(\vec{c} + \lambda \sum_{i=1}^{k} y_i \vec{u}_i) - f(\vec{c} + \lambda \sum_{i=1}^{k-1} y_i \vec{u}_i)$, so if we set

$\vec{b}_k = \vec{c} + \lambda \sum_{i=1}^{k-1} y_i \vec{u}_i$ then we have

$f(\vec{b}_k + \lambda y_k \vec{u}_k) - f(\vec{b}_k)$, where $\vec{b}_k + \lambda y_k \vec{u}_k$ and $\vec{b}_k$ differ only in the k^{th} coordinate. Thinking of $\vec{b}_k$ and $\vec{b}_k + \lambda y_k \vec{u}_k$ as the endpoints of a line segment, by the 1-dim MVT there exists a point $\vec{a}_k$ such that

$$f(\vec{b}_k + \lambda y_k \vec{u}_k) - f(\vec{b}_k) = \lambda y_k D_k f(\vec{a}_k)$$

where $a_k \in L(\vec{b}_k + \lambda y_k \vec{u}_k, \vec{b}_k)$.

Since $\vec{b}_k \longrightarrow \vec{c}$ as $\lambda \to 0$ and $\vec{a}_k \to \vec{c}$ as $\lambda \to 0$,

we can write, since $D_k f$ is continuous for $k \geq 2$:

$$\lim_{\lambda \to 0} D_k f(a_k) = D_k f(\vec{c})$$

or in other words $D_k f(a_k) = D_k f(\vec{c}) + E_k(\lambda)$

where $\lim_{\lambda \to 0} E_k(\lambda) = 0$.

Then

$$f(b_k + \lambda y_k u_k) - f(b_k) = \lambda y_k \left(D_k f(\vec{c}) + E_k(\lambda) \right)$$

and so

$$f(\vec{c}+\vec{v}) - f(\vec{c}) = \lambda \sum_{k=1}^{n} y_k D_k f(\vec{c}) + \lambda \sum_{k=1}^{n} y_k E_k(\lambda)$$

$$= \nabla f(\vec{c}) \vec{v} + \text{Error}$$

and note that $\lim\limits_{\lambda \to 0} \lambda \sum_{k=1}^{n} y_k E_k(\lambda) = 0$.

So the derivative exists, and it is $\nabla f(\vec{c})$.

§12.3 Equality of mixed partials

Since $D_i f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is itself a function of n variables, we can take its k^{th} partial to get a second-order partial denoted

$$D_{k,i} f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

alternatively, $D_{k,i} f = \dfrac{\partial^2 f}{\partial x_k \partial x_i}$

We do not necessarily have $D_{k,i} f(\vec{c}) = D_{i,k} f(\vec{c})$, as this example shows.

Example: Let

$$f(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Then we calculate:

$$D_1 f(x,y) = \frac{y(x^4+4x^2y^2-y^4)}{(x^2+y^2)^2}, \quad (x,y) \neq (0,0)$$ using the ordinary quotient rule. At zero we get

$$D_1 f(0,0) = \lim_{h \to 0} \frac{f(x+h,y) - f(0,0)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)y((x+h)^2-y^2)}{h((x+h)^2+y^2)}$$ with $x=0$, $y=0$ becomes

$$= \lim_{h \to 0} \frac{0}{h^3} = 0.$$

Thus $D_1 f(0,y) = -y$ for all $y \in \mathbb{R}$, and so we can compute $D_{2,1} f(0,y) = -1$ for all $y \in \mathbb{R}$.

On the other hand, if we calculate

$$D_2 f(x,y) = \frac{x(x^4 - 4x^2y^2 - y^4)}{(x^2+y^2)^2} \quad \text{if } (x,y) \neq (0,0)$$

and similarly

$$D_2 f(0,0) = \lim_{h \to 0} \frac{f(0,0+h) - f(0,0)}{h}$$

$$= \lim_{h \to 0} \frac{0}{h(0^2+h^2)} = 0.$$

And therefore $D_2(f(x,0)) = x$ for all $x \in \mathbb{R}$. So we get $D_{1,2} f(x,0) = D_1(x) = 1$ for all $x \in \mathbb{R}$. In particular

$$D_{1,2} f(0,0) = 1 \quad \text{while} \quad D_{2,1} f(0,0) = -1.$$

We'll show:

<u>Theorem</u>: If both partial derivatives $D_r f$ and $D_k f$ exist in some open ball $B(\vec{c})$, and both are differentiable at $\vec{c}$, then

$$D_{r,k} f(\vec{c}) = D_{k,r} f(\vec{c}).$$

Where does our example fail the hypotheses of the theorem? Consider $D_1 f(x,y)$, which we'll denote by $g(x,y)$:

$$g(x,y) = \begin{cases} \dfrac{y(x^4 + 4x^2y^2 - y^4)}{(x^2+y^2)^2} & \text{if } (x,y) \neq (0,0) \\ \cancel{(0,0)}\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

If g were differentiable at $(0,0)$, then it would have directional derivatives in all directions given by the formula:

$$g((0,0);\vec{v}) = \nabla g(0,0) \cdot \vec{v}$$

$$= D_1 g(0,0)\, v_1 + D_2 g(0,0)\, v_2, \quad \text{where } \vec{v} = (v_1, v_2).$$

We computed already that $D_2 g(0,0) = D_2(D_1 f)(0,0) = -1$.

On the other hand,

$$D_1 g(0,0) = \lim_{h \to 0} \frac{0(h^4 + 4h^2 \cdot 0 - 0^4)}{(h^2 + 0^2)^2} = 0.$$

So we must have $g((0,0);\vec{v}) = -v_2$ for all $\vec{v}$.

Consider $\vec{v} = (1,1)$, and compute $g((0,0);\vec{v})$ from the definition:

$$g((0,0);(1,1)) = \lim_{h \to 0} \frac{g(h,h) - \cancel{g(0,0)}^{\,0}}{h}$$

$$= \lim_{h \to 0} \frac{4h^5}{h \cdot (2h^2)^2} = \lim_{h \to 0} 1 = 1,$$

which contradicts the above formula. So our first partials fail to be differentiable at $(0,0)$.

Proof of theorem:

First, we note that if $f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ then by definition if $f = (f_1, \dots, f_m)$ then

$$D_k f = (D_k f_1, D_k f_2, \dots, D_k f_m)$$

and so $D_{i,k} f = D_{k,i} f$ if and only if $D_{i,k} f_j = D_{k,i} f_j$ for $j = 1, \dots, m$. So it suffices to prove the theorem for functions $f: \mathbb{R}^n \longrightarrow \mathbb{R}$.

Moreover, the conclusion we want, namely

$$D_{r,k} f(\vec{c}) = D_{k,r} f(\vec{c})$$

involves only the r^{th} and k^{th} coordinate directions, all others will not affect the steps in our proof (check this as we proceed). Thus we also may assume $n = 2$.

Last, we may also assume $\vec{c} = (0,0)$ by replacing $f: \mathbb{R}^2 \longrightarrow \mathbb{R}$ by "fg": $\mathbb{R}^2 \longrightarrow \mathbb{R}$, where if $\vec{c} = (c_1, c_2)$ then $g(x,y) = (x - c_1, y - c_2)$. Then f and fg have the "same" derivatives, in the sense that they're equal up to applying the chain rule, but $fg(\vec{c}) = f(0,0)$.

So we want to prove, with these simplifications:

$$D_{1,2} f(0,0) = D_{2,1} f(0,0)$$

Choose $h>0$ so that the square $[0,h]\times[0,h]$ is contained in $B(0,0)$, the ball where $D_1 f$ and $D_2 f$ exist. We will consider the limit

$$\lim_{h\to 0} \frac{f(h,h)-f(h,0)-f(0,h)+f(0,0)}{h^2}$$

and show it is equal to <u>both</u> $D_{2,1}f(0,0)$ and $D_{1,2}f(0,0)$, and thus the derivatives are equal.

Set $G(x)=f(x,h)-f(x,0)$ and observe the numerator above is equal to $G(h)-G(0)$. By the MVT there exists $x_1 \in (0,h)$ such that

$$\begin{aligned} G(h)-G(0) &= G'(x_1)\cdot(h-0) \\ &= h\left(D_1 f(x_1,h)-D_1 f(x_1,0)\right). \end{aligned}$$

By assumption $D_1 f:\mathbb{R}^2\to\mathbb{R}$ is differentiable at $(0,0)$, so we have

$$D_1 f(x_1,h) = D_1 f(0,0) + \underbrace{\nabla D_1 f \bullet (x_1,h)}_{\text{derivative evaluated at } (x_1,h)} + (x_1^2+h^2)^{1/2} E_1(h)$$

$$= D_1 f(0,0) + D_{1,1}f(0,0)\,x_1 + D_{2,1}f(0,0)\,h + (x_1^2+h^2)^{1/2}E_1(h)$$

where $\lim_{h\to 0} E_1(h)=0$. Taking the derivative with $(x_1,0)$ ~~only~~, we also have

$$D_1 f(x_1,0) = D_1 f(0,0) + D_{1,1} f(0,0)x_1 + \boxed{0}^{\text{since } h=0} + |x_1| E_2(h)$$

where $\lim_{h\to 0} E_2(h) = 0.$

Now considering the numerator of the limit we're investigating:

$$f(h,h) - f(h,0) - f(0,h) + f(0,0) = G(h) - G(0)$$

$$= h\left(D_1 f(x_1,h) - D_1 f(x,0)\right) \text{ (by MVT)}$$

$$= h\Big(\cancel{D_1 f(0,0)} + \cancel{D_{1,1} f(0,0)x_1} + D_{2,1} f(0,0)h + (x_1^2+h^2)^{1/2} E_1(h)$$

$$-\left(\cancel{D_1 f(0,0)} + \cancel{D_{1,1} f(0,0)x_1} + |x_1| E_2(h)\right)\Big)$$

$= D_{2,1} f(0,0) h^2 + E(h)$, where $E(h)$ is a combination of the error terms above.

We use $|x_1| \le |h|$ to find

$$0 \le |E(h)| = \left| h(x_1^2+h^2)^{1/2} E_1(h) - h|x_1| E_2(h)\right|$$

$$\le \sqrt{2}\, h^2 |E_1(h)| + h^2 |E_2(h)|$$

So
$$\lim_{h\to 0} \frac{f(h,h) - f(h,0) - f(0,h) + f(0,0)}{h^2}$$

$$= \lim_{h\to 0} \frac{D_{2,1} f(0,0) h^2 + E(h)}{h^2} = D_{2,1}(0,0) + \cancel{\lim_{h\to 0} \frac{E(h)}{h^2}}^{\,0},$$

where the last limit uses the formula above.

We can repeat this procedure using $H(y) = f(h,y) - f(0,y)$ in place of $G(x)$, and find

$$\lim_{h \to 0} \frac{\text{top}}{h^2} = D_{1,2} f(0,0),$$

which completes the proof.

<u>Theorem</u> (Combining this theorem and previous one)

If both partials $D_k f$ and $D_r f$ exists in a ball $B(\vec{c})$ and if both $D_{k,r} f$ and $D_{r,k} f$ are continuous at $\vec{c}$, then

$$D_{r,k} f(\vec{c}) = D_{k,r} f(\vec{c}).$$