MATH 3472

By popular request, an example of integral transform applications and convolution.

Sample application: Solving DE's with discontinuous forcing functions. The general scheme is as follows:

① Begin with DE in $y'(t)$, $y''(t)$, $y(t)$ etc.

② Apply the Laplace transform $\mathcal{L}$ to the DE.
Set $Y(s) = \mathcal{L}\{y(t)\}$, and we arrive at an equation with no derivatives in s and $Y(s)$.

③ Solve for $Y(s)$.

④ Apply $\mathcal{L}^{-1}$ to $Y(s)$ to return to $y(t)$.

Steps ② and ④ are generally done by using tables to look up formulas for $\mathcal{L}$, $\mathcal{L}^{-1}$.

<u>Example</u>: Calculate $\mathcal{L}\{e^{at}\}$.

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t)\,dt$$

Therefore

$$\mathcal{L}\{e^{at}\} = \int_0^{\infty} e^{at-st}\,dt$$

$$= \lim_{b\to\infty} \int_0^b e^{t(a-s)}\,dt$$

$$= \lim_{b\to\infty} \left[\frac{1}{a-s} e^{(a-s)t}\right]_0^b = \frac{1}{s-a},$$

as long as $s > a$ so that $\lim_{b\to\infty} e^{(a-s)b} = 0$.

So one entry in the table would be

$f(t)$	$F(s)$
$\longrightarrow$ e^{at}	$\dfrac{1}{s-a}$
t^n	$\dfrac{n!}{s^{n+1}}$
$\sin(at)$	$\dfrac{a}{s^2+a^2}$
$\cos(at)$	$\dfrac{s}{s^2+a^2}$
$e^{at}\sin(bt)$	$\dfrac{b}{(s-a)^2+b^2}$
$e^{at}\cos(bt)$	$\dfrac{s-a}{(s-a)^2+b^2}$

+ many, many more.

We can also compute $\mathcal{L}\{y^{(n)}(t)\}$ for any n.

<u>Example</u>: Compute $\mathcal{L}\{y'(t)\}$.

<u>Solution</u>:

$$\mathcal{L}\{y'(t)\} = \int_0^\infty e^{-st} y'(t)\,dt$$

$$= \left[e^{-st}y(t)\right]_0^\infty - \int_0^\infty y(t)(-se^{-st})\,dt$$

$$= \left[e^{-st}y(t)\right]_0^\infty + s\underbrace{\int_0^\infty y(t)e^{-st}\,dt}_{\mathcal{L}\{y(t)\} = Y(s).}$$

Applying limits to the first term, we get

$\mathcal{L}\{y'(t)\} = -y(0) + sY(s)$.

In general, by induction we get:

$$\boxed{\mathcal{L}\{y^{n}(t)\} = s^{n}Y(s) - s^{n-1}y(0) - s^{n-2}y'(0) - \dots - y^{(n-1)}(0)}$$

One other significant formula before we get started:

$\boxed{\mathcal{L}\{e^{at}y(t)\} = Y(s-a)}$ "shift formula".

Example: Use Laplace transforms to solve

$$y'' - 2y' + y = 2e^{t}, \quad y(0) = y'(0) = 0.$$

Solution: Applying $\mathcal{L}$ to both sides:

$$\mathcal{L}\{y''\} - 2\mathcal{L}\{y'\} + \mathcal{L}\{y\} = 2e^{t}, \quad y(0) = y'(0) = 0.$$

$$\Rightarrow (s^2Y(s) - sy(0) - y'(0)) - 2(sY(s) - y(0)) + Y(s) = \frac{2}{s-1}$$

$$\Rightarrow Y(s)(s^2 - 2s + 1) = \frac{2}{s-1}$$

$$\Rightarrow Y(s) = \frac{2}{(s-1)^3}.$$

Thus

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{2}{(s-1)^3}\right\}$$

formula is: $\frac{1}{s^3}$, but shifted! (Here $a=1$)

use table

$$y(t) = 2e^{t}\mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\} = 2e^{t}\left(\frac{t^2}{2}\right) = \boxed{t^2e^{t}}$$

Laplace transforms also can be used when the RHS is discontinuous. E.g. if

$$h(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 1 \end{cases}$$

then $y(t)\,h(t-a)$ is a function that "turns on" at $t = a$, and

$$\mathcal{L}\{y(t)\,h(t-a)\} = e^{-sa}\,\mathcal{L}\{y(t+a)\}.$$ (Just a mechanical check).

<u>Example</u>: Solve $y'' + y = h(t-2) - h(t-4)$; $y(0) = a$, $y'(0) = b$. (general solution).

<u>Solution</u>:

$$\mathcal{L}\{y''\} + \mathcal{L}\{y\} = \frac{e^{-2s}}{s} - \frac{e^{-4s}}{s}$$

$$\Rightarrow s^2 Y(s) - s\,y(0) - y'(0) - Y(s) = \frac{e^{-2s}}{s} - \frac{e^{-4s}}{s}.$$

$$\Rightarrow Y(s) = \frac{1}{s^2+1}\left(\frac{e^{-2s}}{s} - \frac{e^{-4s}}{s} + as + b\right)$$

.. Now use a variety of algebra tricks to massage this expression so that it looks like a sum of entries in a standard Laplace transform table...

$$\Rightarrow Y(s) = e^{-2s}\left(\frac{1}{s} - \frac{s}{s^2+1}\right) - e^{-4s}\left(\frac{1}{s} - \frac{s}{s^2+1}\right) + \frac{as}{s^2+1} + \frac{b}{s^2+1}$$

Take inverse Laplace..

$$y(t) = h(t-2)\bigl(1 - \cos(t-2)\bigr) - h(t-4)\bigl(1 - \cos(t-4)\bigr) + a\cos t + b\sin t.$$

By far the hardest step is always $\mathcal{L}^{-1}$. There are some things which simply do not appear in tables, so we get stuck – e.g. products. Thankfully:

Theorem: Suppose $\mathcal{L}\{f(t)\} = F(s)$ and $\mathcal{L}\{g(t)\} = G(s)$. Then $\mathcal{L}^{-1}\{F(s)G(s)\} = (f * g)(t) = \int_0^t f(u)g(t-u)\,du$.

So, for example:

$$F(s) = \frac{s}{s^2+1}, \quad G(s) = \frac{1}{s^2+1}.$$ Then

$$\mathcal{L}^{-1}\{F(s)G(s)\} = f * g \quad \text{where } f = \cos(t),\ g = \sin(t)$$

$$= \int_0^t \cos t \sin(t-u)\,du$$

$=$ double integration by parts, trig tricks

$$= \frac{1}{2} t \sin t.$$

This would come up solving something like

$y'' + y = f(t)$ wher $y(0) = y'(0) = 0$ and

$$f(t) = \begin{cases} \cos t & \text{if } 0 < t < \frac{3\pi}{2} \\ \sin t & \text{if } t > \frac{3\pi}{2}. \end{cases}$$

Then taking $\mathcal{L}$ of both sides yields

$$Y(s) = \frac{s}{(s^2+1)^2} - e^{-\frac{3\pi}{2}s}\left(\frac{1}{(s^2+1)^2} + \frac{s}{(s^2+1)^2}\right)$$

The term $\frac{s}{(s^2+1)} = \frac{1}{s^2+1} \cdot \frac{s}{s^2+1}$ requires convolution. In the end, we find:

$$y(t) = \tfrac{1}{2} t \sin t - h\left(t - \tfrac{3\pi}{2}\right)\left(\tfrac{1}{2}\sin\left(t - \tfrac{3\pi}{2}\right) - \left(t - \tfrac{3\pi}{2}\right)\cos\left(t - \tfrac{3\pi}{2}\right)\right)$$

$$- h\left(t - \tfrac{3\pi}{2}\right)\left(\tfrac{1}{2}\left(t - \tfrac{3\pi}{2}\right)\cos(t)\right).$$

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Recall that we saw: One of the essential properties of convolution was its behaviour with respect to products:

$$\mathcal{L}^{-1}\{F(s)G(s)\} = (f * g)(t),$$

or equally

$$\mathcal{L}\{f * g\} = \mathcal{L}\{f\} * \mathcal{L}\{g\}.$$

Here, $\mathcal{L}$ is the Laplace transform from last day.

It should be no surprise that the same is true for the Fourier transform $\mathcal{F}$, since

$$\mathcal{F}(f(x)) = \int_{-\infty}^{\infty} e^{-ixy} f(x)\,dx$$

while $\mathcal{L}(f(x)) = \int_0^{\infty} e^{-xy} f(x)\,dx$

<u>Theorem</u>: Let $f, g \in L(\mathbb{R})$ be given and assume that at least one of f or g is bounded on $\mathbb{R}$ (so we can apply the result from last week to conclude $f * g$ exists $\forall x$) Set $h = f * g$. Then for all $u \in \mathbb{R}$ we have

$$\int_{-\infty}^{\infty} h(x) e^{-ixu} = \left(\int_{-\infty}^{\infty} f(t) e^{-itu}\,dt\right)\left(\int_{-\infty}^{\infty} g(y) e^{-iyu}\,dy\right)$$

ie $\mathcal{F}(f * g) = \mathcal{F}(f) \cdot \mathcal{F}(g)$, same as the Laplace transform.

Proof: WLOG assume g is continuous and bounded on $\mathbb{R}$. Suppose $\{a_n\}$ and $\{b_n\}$ are increasing sequences of positive real numbers with $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \infty$. Define $\{f_n(t)\}$ by

$$f_n(t) = \int_{-a_n}^{b_n} e^{-iux} g(x-t)\, dt .$$

Now since, for every $[a,b]$ we have

$$\int_a^b |e^{-iux} g(x-t)|\, dt \le \int_{-\infty}^{\infty} |g| \quad (\text{since } |e^{-iux}| = 1),$$

Theorem 10.31 gives

$$\lim_{n\to\infty} f_n(t) = \int_{-\infty}^{\infty} e^{-iux} g(x-t)\, dx \quad \text{for every real } t.$$

(Theorem 10.31 was about Lebesgue integrability from the boundedness of integrals on compact subsets)

Then set $y = x-t$ to get $(x = y+t)$

$$\int_{-\infty}^{\infty} e^{-iux} g(x-t)\, dx = \int_{-\infty}^{\infty} e^{-iu(y+t)} g(y)\, dy$$

$$= e^{-iut} \int_{-\infty}^{\infty} e^{-iuy} g(y)\, dy .$$

Then $\lim_{n\to\infty} f_n(t) = e^{-iut} \int_{-\infty}^{\infty} e^{-iuy} g(y)\, dy$, and so

$$\lim_{n\to\infty} f(t) f_n(t) = f(t) e^{-iut} \int_{-\infty}^{\infty} e^{-iuy} g(y)\, dy \quad \text{for all } t.$$

Now f_n is continuous on $\mathbb{R}$ (by Theorem 10.38), and so

$f \cdot f_n$ is a product of a continuous and Lebesgue integrable function, thus measurable on $\mathbb{R}$. Then

$$|f(t) f_n(t)| \leq |f(t)|\,|f_n(t)|$$
$$= |f(t)| \left| \int_{-a_n}^{b_n} e^{-iux} g(x-t)\,dx \right|$$
$$\leq |f(t)| \int_{-\infty}^{\infty} |g| \quad \text{(as we saw above)}$$

and so $f \cdot f_n$ is actually in $L(\mathbb{R})$. Then recall the Lebesgue dominated convergence theorem:

Theorem: Assume $f_n : \mathbb{R} \to \mathbb{R}$ are Lebesgue measurable and $f(x) = \lim_{n\to\infty} f_n(x)$ is the pointwise limit, ~~then~~ and $\exists$ an integrable $g : \mathbb{R} \to [0, \infty]$ with $|f_n(x)| < g(x)\ \forall x \in \mathbb{R}$. Then f is integrable, as is each f_n, and

$$\lim_{n\to\infty} \int_{\mathbb{R}} f_n = \int_{\mathbb{R}} \lim_{n\to\infty} f_n = \int_{\mathbb{R}} f.$$

So we can apply this theorem here to $f(t) f_n(t)$ and get

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f(t) f_n(t)\,dt = \int_{-\infty}^{\infty} \lim_{n\to\infty} f(t) f_n(t)\,dt$$
$$= \int_{-\infty}^{\infty} f(t) e^{-iut} \int_{-\infty}^{\infty} e^{-iuy} g(y)\,dy\,dt$$
$$= \left(\int_{-\infty}^{\infty} f(t) e^{-iut}\,dt\right) \cdot \left(\int_{-\infty}^{\infty} e^{-iuy} g(y)\,dy\right)$$
$$\mathcal{F}(f(t)) \cdot \mathcal{F}(g(t))$$

On the other hand, we can also compute

$$\int_{-\infty}^{\infty} f(t) f_n(t)\,dt = \int_{-\infty}^{\infty} f(t) \left[\int_{-a_n}^{b_n} e^{-iux} g(x-t)\,dx \right] dt$$

But now $k(x,t) = g(x-t)$ is continuous and bounded on $\mathbb{R}^2$ and $\int_a^b e^{-iux}\,dx$ exists $\forall\, [a,b] \subseteq \mathbb{R}$, so Theorem 10.40 applies and we can reverse the order of integration:

$$\int_{-\infty}^{\infty} f(t) f_n(t)\,dt = \int_{-a_n}^{b_n} e^{-iux} \left[\int_{-\infty}^{\infty} f(t) g(x-t)\,dt \right] dx$$

$$= \int_{-a_n}^{b_n} h(x) e^{-iux}\,dx.$$

Thus by combining with previous inequalities

$$\underbrace{\lim_{n\to\infty} \int_{a_n}^{b_n} h(x) e^{-iux}\,dx}_{= \mathcal{F}(h(x))} = \mathcal{F}(f) \cdot \mathcal{F}(g)$$

Remark: As a special case, this proves the Laplace transform convolution theorem from last day.

Since $\mathcal{L}(f) = \int_0^{\infty} e^{-xy} f(x)\,dx$ we can apply the above theorem with f and g zero to the left of zero.

Application example:

Solve the integral equation $f(t) = 2\cos t - \int_0^t (t-u) f(u)\,du$.

Solution: Applying the Laplace transform $\mathcal{L}$, write $F(s) = \mathcal{L}\{f(t)\}$ and use $\mathcal{L}\{t\} = \frac{1}{s^2}$. Then

$$F(s) = \frac{2s}{s^2+1} - \mathcal{L}\{t * f(t)\}$$

$$= \frac{2s}{s^2+1} - \mathcal{L}\{t\} \cdot \mathcal{L}\{f(t)\}$$

$$= \frac{2s}{s^2+1} - \frac{1}{s^2} \cdot F(s)$$

$$\Rightarrow F(s)\left(1 + \frac{1}{s^2}\right) = \frac{2s}{s^2+1}$$

$$\Rightarrow F(s) = \frac{2s^3}{(s^2+1)^2} = \frac{2s}{s^2+1} - \frac{2s}{(s^2+1)^2}.$$ use tables to do $\mathcal{L}^{-1}$, we get

$$f(t) = \mathcal{L}^{-1}(F(s)) = \mathcal{L}^{-1}\left(\frac{2s}{s^2+1}\right) - \mathcal{L}^{-1}\left(\frac{2s}{(s^2+1)^2}\right)$$

$$= 2\cos t - t\sin t.$$