# MATH 3472

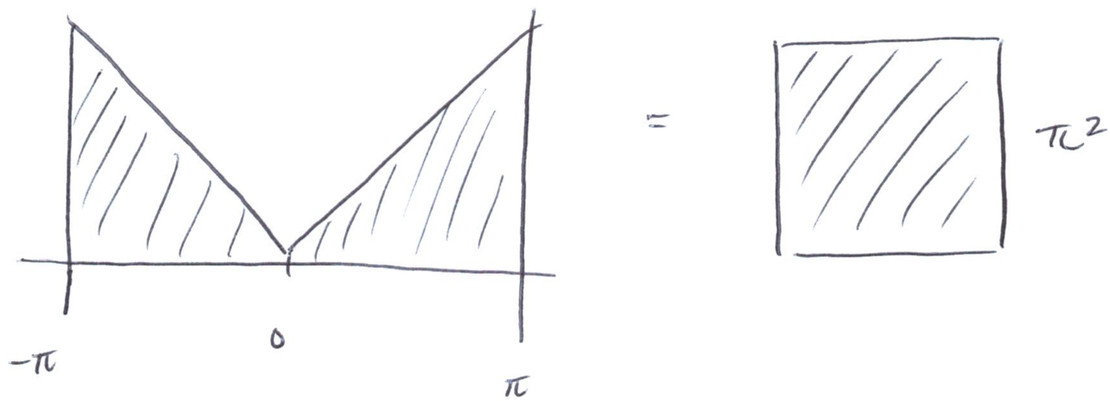
**Example:** Suppose $f(x) = |x|$, $-\pi \le x \le \pi$.

Then $f(x)$ is even, so if

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx)$$

then $b_k = 0$. For $a_0$ we compute

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\,dx = \pi, \text{ and}$$




for $n \ge 1$
$$a_n = \frac{2}{\pi} \int_0^{\pi} x \cos nx\,dx$$

$$= \underbrace{\frac{2}{\pi} x \frac{\sin nx}{n} \Big|_0^{\pi}}_{0} - \frac{2}{\pi} \int_0^{\pi} \frac{\sin nx}{n}\,dx$$

$$= + \frac{2}{n\pi} \frac{\cos nx}{n} \Big|_0^{\pi} = \frac{2}{n^2\pi}\left((-1)^n - 1\right) = \begin{cases} 0 & n \text{ even} \\ \frac{-4}{\pi n^2} & n \text{ odd} \end{cases}$$

So $|x| \sim \frac{\pi}{2} - \frac{4}{\pi}\cos x - \frac{4}{\pi 3^2}\cos 3x - \frac{4}{\pi 5^2}\cos 5x - \cdots$,

and this converges on $[-\pi, \pi]$ to $f(x)$ since $f$ is cts and BV.

So when $x=0$, e.g. we get

$$0 = \frac{\pi}{2} - \frac{4}{\pi} - \frac{4}{\pi 3^2} - \frac{4}{\pi 5^2} - \cdots$$

$$\Rightarrow \frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots$$

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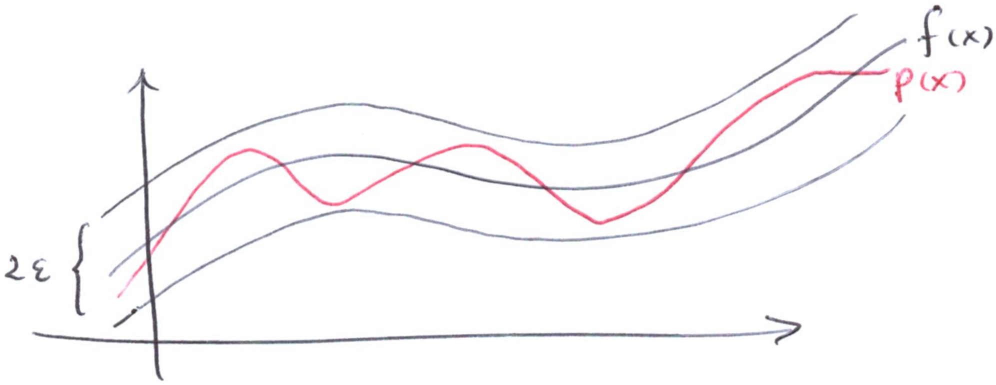
We can also use Fejér's Theorem to provide a quicker proof of some famous theorems, e.g. Weierstrass' approx. Theorem

<u>Theorem 11.17</u>: Let $f(x)$ be real-valued and continuous on $[a,b]$. Then $\forall \varepsilon > 0$ $\exists$ a polynomial $p(x)$ such that $|f(x) - p(x)| < \varepsilon$ $\forall x \in [a,b]$.

i.e.




"Every continuous function on a compact interval is a uniform limit of polynomials"

<u>Proof</u>: For $t \in [0, \pi)$ set

$g(t) = f\left(a + \frac{t(b-a)}{\pi}\right)$ and for $t \in [\pi, 2\pi]$ set

$g(t) = f\left(a + \frac{(2\pi - t)(b-a)}{\pi}\right)$, and extend the function

$g(t)$ $2\pi$-periodically to all of $\mathbb{R}$. Here, $g(t)$ is is a reparameterization of $f(x)$ that changes the domain to $[0, 2\pi]$ and extends periodically. Thus Fejèr's theorem applies: There is a function $\sigma(t)$ defined by a formula of the form

$$\sigma(t) = A_0 + \sum_{k=1}^{N} A_k \cos kt + B_k \sin kt$$

such that $|g(t) - \sigma(t)| < \varepsilon/2 \quad \forall t \in [0, 2\pi]$ (Here, $\sigma(t)$ is one of the functions $\sigma_n$ which converge uniformly to $f$ on $[0, 2\pi]$, $\sigma_n$'s arising from Cesàro summability of the Fourier series).

The function $\sigma(t)$ is a finite sum of trig functions, thus it admits a power series that converges uniformly on every compact interval about the origin. Let $p_n(t)$ denote the $n^{th}$ partial sum of the power series of $\sigma(t)$. Then $\{p_n(t)\}$ is a sequence of polynomials converging uniformly to $\sigma(t)$ on $[0, 2\pi]$.

Thus for same $\varepsilon$ as above $\exists$ $m$ s.t.

$$|p_m(t) - \sigma(t)| < \tfrac{\varepsilon}{2} \quad \forall t \in [0, 2\pi].$$

Thus $|p_m(t) - g(t)| < \varepsilon \quad \forall t \in [0, 2\pi]$.

Now set $p(x) = p_m\left(\frac{\pi(x-a)}{b-a}\right)$. Then with $t = \frac{\pi(x-a)}{b-a}$

$$|p_m(t) - g(t)| =$$

$$\left|p_m\left(\frac{\pi(x-a)}{b-a}\right) - g\left(\frac{\pi(x-a)}{b-a}\right)\right|$$

$$= \left|p(x) - f\left(a + \left(\frac{\pi(x-a)}{b-a}\right)\left(\frac{b-a}{\pi}\right)\right)\right| \quad (\text{if } t \in [0,\pi))$$

$$= |p(x) - f(a+x-a)|$$

$= |p(x) - f(x)|$, which is $< \varepsilon$. Similarly for $t \in [\pi, 2\pi]$.

Thus the Weierstrass approx. Theorem holds.

Alternative, more informative proof:

Having arrived at

$$\sigma(t) = \sum_{k=0}^{N} (\underline{\quad \text{stuff,} \quad})$$ it would be nicer if this were <u>already</u> a polynomial.

Alternatively set $g(t) = f(x)$, where $\cos(t) = x$, $x \in [-1,1]$ and $t \in [0,\pi]$ (Here we assume $[a,b] = [-1,1]$ by scaling if necessary). Extend to $t \in [-\pi, 0]$ using the formula

$$g(-t) := g(t),$$

and arrive at $\sigma_N = \sum_{k=0}^{N} \gamma_k \cos kt$ using Fejèr's theorem as before. Note there are no sines since $g(t)$ is even.

Now since $t = \arccos(x)$ we've got, from $|g(t) - \sigma(t)| < \varepsilon$:

$$|f(x) - \sigma_N(\arccos(x))| < \varepsilon$$

But then $\sigma_N(\arccos x)$ is a polynomial!

This follows from

Theorem: For every positive integer $k$,

$$\cos(k \arccos x) \text{ is a polynomial } (x \in [-1,1]).$$

Proof: There is the following identity for cosines:

$$\cos((n+1)\theta) = 2\cos(\theta)\cos(n\theta) - \cos((n-1)\theta) \qquad (*)$$

which one can prove from

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

using $\alpha = n\theta$, $\beta = \theta$ and $\beta = -\theta$. Add resulting equations, rearrange.

Use $(*)$ and induction to establish the existence of $c_i$ such that

$$\cos n\theta = \sum_{k=0}^{n} c_k \cos^k(\theta).$$

Setting $x = \cos\theta$, (ie $\theta = \arccos x$ for $x \in [-1,1]$) this becomes

$$T_n(x) := \cos(n \arccos x) = \sum_{k=0}^{n} c_k x^k, \text{ a polynomial.}$$

These are the Chebyshev polynomials of the first kind.

## § 11.16 Other forms of Fourier series.

It's worth noting that there are other orthonormal families, and other ways of writing the series we've found so far.

We already saw that if $f(x)$ is $p$-periodic for some $p$, then $\cos(nx)$, $\sin(nx)$ are replaced by $\cos\left(\frac{2\pi nx}{p}\right)$, $\sin\left(\frac{2\pi nx}{p}\right)$ and for $f \in L([0,p])$

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty}\left(a_k \cos\left(\frac{2\pi kx}{p}\right) + b_k \sin\left(\frac{2\pi kx}{p}\right)\right)$$

where $$a_k = \frac{2}{p}\int_0^p f(t)\cos\left(\frac{2\pi kt}{p}\right)dt$$

$$b_k = \frac{2}{p}\int_0^p f(t)\sin\left(\frac{2\pi kt}{p}\right)dt .$$

Or $$f(x) \sim \sum_{n=-\infty}^{\infty} a_n e^{\frac{2\pi inx}{p}}, \text{ where}$$

$$a_n = \frac{1}{p}\int_0^p f(t)e^{-\frac{2\pi int}{p}}dt .$$

All work done so far can be appropriately reformulated to work with these series.

§ 11.17 : The Fourier Integral theorem.

The motivation here is as follows: Only when $f$ is periodic on $(-\infty, \infty)$ can we hope to find a Fourier series that converges to $f$ on $(-\infty, \infty)$.

We can drop the periodicity condition if we ask for an integral representation of $f(x)$ instead of a representation as a sum.

**Theorem** (Fourier Integral Theorem)

Suppose $f \in L(\mathbb{R})$ and $\exists\, x \in \mathbb{R}$ and $\delta > 0$ such that either

(i) $f$ has BV on $[x-\delta, x+\delta]$

(ii) $\lim_{t \to x^-} f(t) = f(x^-)$ and $\lim_{t \to x^+} f(t) = f(x^+)$ both exist

and both integrals

$$\int_0^\delta \frac{f(x+t) - f(x+)}{t}\, dt, \quad \int_0^\delta \frac{f(x-t) - f(x-)}{t}\, dt$$

exist.

Then
$$\frac{1}{\pi}\int_0^\infty \left[\int_{-\infty}^\infty f(u) \cos v(u-x)\, du\right] dv = \frac{f(x+) + f(x-)}{2},$$

where $\int_0^\infty$ is an improper Riemann integral.

Proof: First we show

$$\lim_{\alpha \to \infty} \frac{1}{\pi}\int_{-\infty}^\infty f(x+t)\frac{\sin \alpha t}{t}\, dt = \frac{f(x+) - f(x-)}{2}$$

To do this, break the integral into pieces:

$$\int_{-\infty}^{\infty} f(x+t)\frac{\sin(at)}{\pi t}\,dt = \int_{-\infty}^{\delta} + \int_{-\delta}^{0} + \int_{0}^{\delta} + \int_{\delta}^{\infty}$$

① ② ③ ④.

Now if we consider the limit as $\alpha\to\infty$, we see $\lim_{\alpha\to\infty}$ ① $= \lim_{\alpha\to\infty}$ ④ $= 0$, both by the Riemann-Lebesgue Lemma.

To deal with ③, we apply Jordan's Theorem if (i) is satisfied, or Dini's Theorem if (ii), and conclude

$$\lim_{\alpha\to\infty}\int_0^{\delta} f(x+t)\frac{\sin\alpha t}{\pi t}\,dt = \frac{f(x+)}{2},$$ in the same way

$$\lim_{\alpha\to\infty}\int_{-\delta}^{0} f(x+t)\frac{\sin\alpha t}{\pi t}\,dt = \lim_{\alpha\to\infty}\int_0^{\delta} f(x-t)\frac{\sin\alpha t}{\pi t}\,dt = \frac{f(x-)}{2}$$

(here we replace $t$ with $-t$/flip the limits).

So our claimed formula holds.

Now $$\int_{-\infty}^{\infty} f(x+t)\frac{\sin\alpha t}{t}\,dt = \int_{-\infty}^{\infty} f(u)\frac{\sin(\alpha(u-x))}{u-x}\,du$$

and since $$\frac{\sin(\alpha(u-x))}{u-x} = \int_0^{\alpha}\cos(v(u-x))\,dv,$$ the limit we established becomes

$$\lim_{\alpha\to\infty}\frac{1}{\pi}\int_{-\infty}^{\infty} f(u)\left[\int_0^{\alpha}\cos(v(u-x))\,dv\right]du = \frac{f(x+)+f(x-)}{2}.$$

This is almost what we seek, but the order of integration is reversed. Then Theorem 10.40 of the book lets us change the order of integration.

The Fourier integral theorem comes into play in our discussion of transforms, which will follow soon. First we mention:

§11.18

Theorem 11.19 : If $f$ satisfies the hypotheses of the Fourier integral theorem, then

$$\frac{f(x+)+f(x-)}{2} = \frac{1}{2\pi} \lim_{\alpha\to\infty} \int_{-\alpha}^{\alpha} \left[ \int_{-\infty}^{\infty} f(u)\, e^{iv(u-x)}\, du \right] dv$$

Proof : Basic idea is to use

$$e^{iv(u-x)} = \underbrace{\cos(v(u-x))}_{\text{apply orginal theorem here,}} + i\sin(v(u-x))$$

and observe

$$\int_{-\alpha}^{\alpha} \Big( \underbrace{\int_{-\infty}^{\infty} f(u)\sin(v(u-x))\, du}_{\substack{\text{odd function} \\ \text{of } v}} \Big) dv = 0$$

§11.19 Integral Transforms.

Idea: Take $f(x) \longmapsto g(y)$ via the process of integration:

$$g(y) = \int_{-\infty}^{\infty} K(x,y) f(x)\,dx,$$ where

$K(x,y)$ is called the <u>kernel</u> of the transform.

Extremely useful, you've probably already seen Laplace transforms:

$$\mathcal{L}\{f\} = \int_0^\infty e^{-xy} f(x)\,dx$$

used in solving DE's with discontinuous forcing, for example.

Also have:

Exponential Fourier Transform $\displaystyle\int_{-\infty}^{\infty} e^{-ixy} f(x)\,dx$

Fourier cosine Transform $\displaystyle\int_0^\infty \cos(xy) f(x)\,dx$

" sine " $\displaystyle\int_0^\infty \sin(xy) f(x)\,dx$

Mellin Transform $\displaystyle\int_0^\infty x^{y-1} f(x)\,dx.$

Notation:

$$\mathcal{F}(f) = g(y), \text{ where } g(y) = \int_{-\infty}^{\infty} e^{-ixy} f(x)\,dx$$

Then <u>remark</u>: $\mathcal{F}$ or $\mathcal{L}$ are often called <u>integral operators</u>, as they are very obviously linear operators:

$\mathcal{F}(af + bg) = a\mathcal{F}(f) + b\mathcal{F}(g)$, by elementary properties of integration.

The exponential Fourier integral theorem is related to $\mathcal{F}$. If we write

$$\mathcal{F}(f) = g(u) = \int_{-\infty}^{\infty} f(t)\, e^{-itu}\, dt,$$

then when $f$ is continuous the Fourier integral theorem gives

$$f(x) = \frac{1}{2\pi} \lim_{\alpha\to\infty} \int_{-\alpha}^{\alpha} \left[ \int_{-\infty}^{\infty} f(u) e^{iv(u-x)}\, du \right] dv$$

$$= \frac{1}{2\pi} \lim_{\alpha\to\infty} \int_{-\alpha}^{\alpha} g(v)\, e^{ixv}\, dv,$$

called the <u>inversion formula</u> for Fourier transforms.

It shows that $f$ is uniquely determined by $\mathcal{F}(f) = g$, whenever $f$ is continuous and satisfies the hypotheses of the Fourier integral theorem. For this reason we sometimes write

$$\mathcal{F}^{-1}(g) = \frac{1}{2\pi} \lim_{\alpha\to\infty} \int_{-\alpha}^{\alpha} g(v)\, e^{ixv}\, dv$$

so that $\mathcal{F}(f) = g$ and $\mathcal{F}^{-1}(g) = f$, as one would hope.

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## §11.20 Convolutions.

The convolution of two functions $f, g$ is a transform of $f$ with a kernel $K(x,y)$ that depends only on the difference $x-y$ and the function $g$.

<u>Definition</u>: Let $f, g$ be Lebesgue integrable on $(-\infty, \infty)$. Let $S \subseteq \mathbb{R}$ denote the set of $x$ for which

$$h(x) = \int_{-\infty}^{\infty} f(t)\, g(x-t)\, dt \text{ exists.}$$

Then $h: S \longrightarrow \mathbb{R}$ is a function called "the convolution of $f$ and $g$", we write $h = f * g$.

<u>Remark</u>: Convolution is commutative: $f*g = g*f$ by a change of variables.

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Special case of importance: Suppose that both $f$ and $g$ are zero $\forall x < 0$.

Then $g(x-t) = 0$ if $t > x$, so we can calculate

$$\int_{-\infty}^{\infty} f(t) g(x-t) dt = \int_0^x f(t)\, g(x-t)\, dt,$$

So $h(x) = (f*g)(x)$ is defined on all points in $[0,b]$ if both $f$ and $g$ are Riemann integrable on $[0,b]$.

(The product of Riemann int functions is Riemann int)

This is <u>not</u> true for Lebesgue integrable functions.

Set $f(t) = \frac{1}{\sqrt{t}}$, $g(t) = \frac{1}{\sqrt{1-t}}$ for $0 < t < 1$.

Then set $f(t) = g(t) = 0 \quad \forall t \notin [0,1)$.

Then

$$\int_{-\infty}^{\infty} f(t)\,dt = \int_0^1 \frac{1}{\sqrt{t}}\,dt = \lim_{\alpha \to 0} \int_\alpha^1 \frac{1}{\sqrt{t}}\,dt$$

$$= \lim_{\alpha \to 0} \Big[ 2\sqrt{t} \Big]_\alpha^1$$

$$= \lim_{\alpha \to 0} 2(1 - \sqrt{\alpha}) = 2, \text{ and}$$

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{1-t}}\,dt = \int_0^1 \frac{1}{\sqrt{1-t}}\,dt = \lim_{\alpha \to 1} \int_0^\alpha \frac{1}{\sqrt{1-t}}\,dt \text{ (set } u = 1-t)$$

$$= \lim_{\alpha \to 0} \int_\alpha^1 \frac{1}{\sqrt{u}}\,du = 2$$

However,

$$\int_{-\infty}^{\infty} f(t)g(1-t)\,dt = \int_0^1 \frac{1}{\sqrt{t}} \cdot \frac{1}{\sqrt{t}}\,dt = \int_0^1 \frac{1}{t}\,dt,$$

which does not exist. The effect here has been that the discontinuity of $f(t)$ at $t=0$ and $g(t)$ at $t=1$ have combined in a "multiplicative way".

When $h(x) = (f*g)(x)$ fails to exist despite both $f$ and $g$ being Lebesgue integrable on an interval containing $x$, we call $x$ a "singularity of $h$".

**Theorem**: Assume $f, g \in L(\mathbb{R})$ and at least one of either $f$ or $g$ is bounded. Then $h(x) = (f * g)(x)$ exists, and if the bounded function ($f$ or $g$) is also continuous, then so is $h$, and $h \in L(\mathbb{R})$.

**Proof**: Since $f*g = g*f$, we can choose that either $f$ or $g$ be the bounded function. Suppose it is $g$, say $|g(t)| \le M$.

Then
$$|f(t)g(x-t)| \le M|f(t)|$$

Theorem 10.35 shows that the integral
$$h(x) = \int_{-\infty}^{\infty} f(t)g(x-t)\,dt \text{ exists for each } x.$$

| |
|---|
| **Theorem** 10.35: If $f \in M(I)$ and $\lvert f(x)\rvert \le g(x)$ for some nonnegative $g(x)$ with $g \in L(I)$, then $f \in L(I)$. |

So here, we check that $f(t)g(x-t)$ is measurable for each $x$ (product of measurable functions). Then use the bound above.

Then we also find
$$|h(x)| = \left| \int_{-\infty}^{\infty} f(t)g(x-t)\,dt \right| \le \int_{-\infty}^{\infty} |f(t)g(x-t)|\,dt$$
$$< M \int_{-\infty}^{\infty} |f(t)|\,dt$$

So $h(x)$ is bounded.

Continuity also follows from a prior theorem:

**Theorem 10.40** (a) Suppose $X, Y \subset \mathbb{R}$ are subintervals, and that $k(x,y)$ is continuous and bounded on $X \times Y$. If $f \in L(X)$ ~~and~~ then

$$F(y) = \int_X f(x)\, k(x,y)\, dx$$

is continuous on $Y$.

So here, we use $g(x-t)$ in place of $k(x,y)$.

Last we check $h(x) \in L(\mathbb{R})$, by using Theorem 10.31.

**Theorem 10.31**: Suppose $f$ is defined on $I$ and that for each $[a,b] \subset I$ there is a positive constant $M$ such that

$$\int_a^b |f| \le M.$$

Then $f \in L(I)$.

We compute:

$$\int_a^b |h(x)|\, dx \le \int_a^b \left| \int_{-\infty}^{\infty} f(t)\, g(x-t)\, dt \right| dx$$

$$= \int_{-\infty}^{\infty} |f(t)| \left[ \int_a^b |g(x-t)|\, dx \right] dt$$

$$= \int_{-\infty}^{\infty} |f(t)| \left[ \int_{a-t}^{b-t} |g(y)|\, dy \right] dt$$

$$\le \int_{-\infty}^{\infty} |f(t)|\, dt \cdot \int_{-\infty}^{\infty} |g(y)|\, dy ;$$

So it follows that $h \in L(I)$.

Therem 11.22: Assume $f \in L^2(\mathbb{R})$ and $g \in L^2(\mathbb{R})$. Then $h(x) = (f * g)(x)$ exists $\forall x \in \mathbb{R}$ and $h(x)$ is bounded.

Proof: Fix $x \in \mathbb{R}$, set $g_x(t) = g(x-t)$. Then $g_x$ is measurable and $g_x \in L^2(\mathbb{R})$, so by Theorem 10.54 the product

$$f \cdot g_x \in L(\mathbb{R}).$$

Thus $h(x) = \int_{-\infty}^{\infty} f g_x \, dt$ exists. Now also note that

$h(x) = (f, g_x)$, and so the Cauchy-Schwartz inequality gives

$$|h(x)| = |(f, g_x)| \leq \|f\| \, \|g_x\| = \|f\| \, \|g\|,$$

so $h(x)$ is bounded.