

Test 1

50 minutes

Name:

Solutions.

Student ID:

Q1]... [10 points] Show that the function

$$f(x, y) = \begin{cases} \frac{x^2 y}{x^4 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

is not differentiable at  $(0, 0)$ , but that all its directional derivatives exist there.

Consider  $\lim_{(x, y) \rightarrow (0, 0)} f(x, y)$ , approaching  $(0, 0)$  along the curve  $(t, t^2)$ . Then

$$\lim_{t \rightarrow 0} \frac{t^2 \cdot t^2}{t^4 + t^4} = \lim_{t \rightarrow 0} \frac{t^4}{2t^4} = \frac{1}{2}$$

So  $f(x, y)$  cannot be continuous at  $(0, 0)$  since  $f(0, 0) = 0$ . Thus it is not differentiable there, since differentiable  $\Rightarrow$  cts.

Now, let  $\vec{v} = (v_1, v_2)$  be an arbitrary nonzero vector and consider

$$\begin{aligned} f'((0, 0); \vec{v}) &= \lim_{h \rightarrow 0} \frac{f((0, 0) + h(\vec{v}_1, \vec{v}_2)) - f(0, 0)}{h} \\ &= \lim_{h \rightarrow 0} \frac{h^3 v_1^2 v_2}{h(h^4 v_1^4 + h^2 v_2^2)} \\ &= \lim_{h \rightarrow 0} \frac{v_1^2 v_2}{h^2 v_1^4 + v_2^2} = \frac{v_1^2}{v_2}, \quad \underline{\underline{\text{provided } v_2 \neq 0.}} \end{aligned}$$

If  $v_2 = 0$ , then the limit is

$$\lim_{h \rightarrow 0} \frac{0}{h(\text{etc.})} = 0.$$

So all directional derivatives exist at  $(0, 0)$ .

Q2]... [10 points]

Show that the function

$$f(x, y) = \begin{cases} (x^2 + y^2) \sin\left(\frac{1}{x^2 + y^2}\right) & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

is differentiable everywhere. (You may use without proof that compositions/sums/products/differences of rational functions and trig functions are continuous on their domain).

To show the function is differentiable at  $(x, y) \neq (0, 0)$ , it suffices to show the partials are continuous at  $(x, y)$ . We compute

$$\frac{\partial f}{\partial x} = 2x \sin\left(\frac{1}{x^2 + y^2}\right) + (x^2 + y^2) \cos\left(\frac{1}{x^2 + y^2}\right) \cdot \frac{-1}{(x^2 + y^2)^2} \cdot 2x$$

and

$$\frac{\partial f}{\partial y} = 2y \sin\left(\frac{1}{x^2 + y^2}\right) + (x^2 + y^2) \cos\left(\frac{1}{x^2 + y^2}\right) \cdot \frac{-1}{(x^2 + y^2)^2} \cdot 2y.$$

These functions are continuous on their domains, which is  $\mathbb{R} \setminus \{(0, 0)\}$ . Thus (since their domain is open) this implies that  $f$  is differentiable at  $(x, y)$ , by a theorem from class.

At  $(0, 0)$ , we use claim  $f'(0, 0)$  exists and it is zero. To see this, consider the Taylor formula with  $\vec{c} = (0, 0)$  and  $\vec{v} = (v_1, v_2)$ :

$$f((0, 0) + (v_1, v_2)) = f(0, 0) + \underbrace{0}_{\text{derivative}} + \|(v_1, v_2)\| E(\vec{v})$$

$$\Rightarrow (v_1^2 + v_2^2) \sin\left(\frac{1}{v_1^2 + v_2^2}\right) = \|(v_1, v_2)\| E(\vec{v}). \text{ We must show that such an } E(\vec{v}) \text{ exists, and } E(\vec{v}) \rightarrow 0 \text{ as } \vec{v} \rightarrow 0.$$

$$\text{Take } E(\vec{v}) = \frac{(v_1^2 + v_2^2) \sin\left(\frac{1}{v_1^2 + v_2^2}\right)}{\|(v_1, v_2)\|} = \frac{(v_1^2 + v_2^2) \sin\left(\frac{1}{v_1^2 + v_2^2}\right)}{\sqrt{v_1^2 + v_2^2}} = \sqrt{v_1^2 + v_2^2} \sin\left(\frac{1}{v_1^2 + v_2^2}\right)$$

$$\text{Now } \lim_{\vec{v} \rightarrow 0} \sqrt{v_1^2 + v_2^2} \sin\left(\frac{1}{v_1^2 + v_2^2}\right) = 0, \text{ since } \sin\left(\frac{1}{v_1^2 + v_2^2}\right) \text{ is bdd and } \sqrt{v_1^2 + v_2^2} \rightarrow 0.$$

$\Rightarrow f$  is differentiable.

Q3)...[10 points] (a) Suppose that  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  satisfies  $f'(x, y) = 0$  for all  $(x, y)$  in  $S^1 = \{(x, y) \mid x^2 + y^2 = 1\}$ . Prove that  $f$  is constant on  $S^1$ .

Let  $g: \mathbb{R} \rightarrow \mathbb{R}^2$  be  $g(t) = (\cos t, \sin t)$ . Then the image of  $g$  is  $S^1$ . The composition  $h = f \circ g$  has derivative

$$\begin{aligned} h'(t) &= f'(\cos t, \sin t) \cdot g'(t) \\ &= 0 \cdot g'(t) = 0. \end{aligned}$$

Thus, as  $h$  has derivative zero on  $\mathbb{R}$ ,  $h$  is constant.

Thus  $f$  must be constant on image of  $g = S^1$ .

(b) Give an example of an open set  $S \subset \mathbb{R}^2$  and a continuous function  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  such that  $f'(c) = 0$  for all  $c \in S$ , yet  $f$  is not constant on  $S$ .

Let  $S = \{(x, y) \mid x < 0\} \cup \{(x, y) \mid x > 1\}$ .

Let  $f(x, y)$  be the function:

$$f(x, y) = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } 0 < x < 1 \\ 1 & \text{if } x > 1. \end{cases}$$

Then  $f$  is continuous, yet on the given open set  $f'$  is zero as the function is constant there.

Q4)... [10 points] Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be a function whose partial derivatives are differentiable everywhere. For a fixed  $\mathbf{v} \in \mathbb{R}^n$ , define a new function  $f_{\mathbf{v}} : \mathbb{R}^n \rightarrow \mathbb{R}$  by  $f_{\mathbf{v}}(\mathbf{x}) = f'(\mathbf{x}; \mathbf{v})$ . Prove or disprove:  $f'_{\mathbf{v}}(\mathbf{x}; \mathbf{u}) = f'_{\mathbf{u}}(\mathbf{x}; \mathbf{v})$  for all  $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ .

We compute  $f_{\mathbf{v}}(\bar{\mathbf{x}})$  using the formula for directional derivatives:

$$\begin{aligned} f_{\mathbf{v}}(\bar{\mathbf{x}}) &= f'(\bar{\mathbf{x}}; \vec{v}) = \nabla f(\bar{\mathbf{x}}) \cdot \vec{v} \\ &= \sum_{i=1}^n \frac{\partial f}{\partial x_i}(\bar{\mathbf{x}}) v_i, \text{ if } \vec{v} = (v_1, \dots, v_n). \end{aligned}$$

Now to compute  $f'_{\mathbf{v}}(\bar{\mathbf{x}}; \vec{u})$  we apply the directional derivative formula again.

$$\begin{aligned} f'_{\mathbf{v}}(\bar{\mathbf{x}}; \vec{u}) &= \nabla \left( \sum_{i=1}^n \frac{\partial f}{\partial x_i}(\bar{\mathbf{x}}) v_i \right) \\ &= \sum_{j=1}^n \frac{\partial}{\partial x_j} \left( \sum_{i=1}^n \frac{\partial f}{\partial x_i}(\bar{\mathbf{x}}) v_i \right) u_j \\ &= \sum_{j=1}^n \sum_{i=1}^n \frac{\partial^2 f}{\partial x_j \partial x_i}(\bar{\mathbf{x}}) v_i u_j \\ &= \sum_{i=1}^n \left( \sum_{j=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j}(\bar{\mathbf{x}}) u_j \right) v_i \end{aligned}$$

by equality of mixed partials, since the first order partials are differentiable

= by the same steps as above, but in reverse...

$$= f'_{\mathbf{u}}(\bar{\mathbf{x}}; \vec{v}).$$