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§0.5 continued

Next we investigate how the rational numbers $\mathbb{Q}$ sit inside of $\mathbb{R}$. To do this, we recall some terminology and make a new definition:

Recall that if $a<b$ are both real numbers, then (a,b) is called an *open* interval and $[a,b]$ is called a closed interval. We define:

Definition: A subset $S \subseteq \mathbb{R}$ is called *dense* if every open interval contains an element of S.

We make this definition now because of the following theorem:

Theorem: The subset $\mathbb{Q} \subseteq \mathbb{R}$ is dense.

Another way of saying the same thing: Between any two real numbers there is a rational number.

Proof: Take any two real numbers x and y with $x<y$. By a theorem from last day, there's a number (integer) N with

$$N-1 \leq \frac{1}{y-x} < N.$$

In particular, since $y-x>0$ we know $\frac{1}{y-x}>0$ and so this integer N gives us $0<\frac{1}{y-x}<N$.

Therefore $0 < 1 < (y-x)N$ also, which we save for future use.

Now use the same theorem to choose a second integer n with $n \le Nx < n+1$. Then we calculate

$$n+1 \le Nx+1 < Nx + N(y-x) \quad (\text{since } 1 < (y-x)N)$$
$$= Ny$$

So we've arrived at two integers n and N satisfying

$$Nx < n+1 < Ny$$

Since $N > 0$

$$x < \frac{n+1}{N} < y.$$

So $\frac{n+1}{N}$ is an element of $\mathbb{Q}$ between x and y.

This implies something that sounds stronger: Between any two real numbers x and y, there are <u>infinitely</u> <u>many</u> rational numbers.

<u>Proof</u>: If $x < y$ are real numbers, choose a rational number r_1 with $x < r_1 < y$. Now choose a rational number r_2 with $r_1 < r_2 < y$. Now choose r_3 with ... etc.

We will soon see another example of a dense subset of $\mathbb{R}$ — the irrational numbers, namely $\mathbb{R} \setminus \mathbb{Q}$.

To prove this fact, we need to know that some irrational numbers exist. We have not yet proved such a thing: We showed that if a number x satisfies $x^2 = 2$ (we denoted x by $\sqrt{2}$) then it cannot be rational. This does not mean that there exists a number x with $x^2 = 2$, so that is what we'll prove now.

<u>Theorem</u>: If p is any positive real number, then there exists a positive real number x such that $x^2 = p$.

<u>Proof</u>: We will do this proof in two cases.

<u>Case 1</u>: Suppose $p \geq 1$. Set

$$A = \{ z \mid z \in \mathbb{R} \text{ is positive and } z^2 \leq p \}.$$

The set A is nonempty since $1 \in A$ and it's bounded above, an upper bound for A is the number p. To see this, note that if $z > p$ then

$z^2 > p^2 \geq p$ (since $p \geq 1$), so $z \notin A$.

Now by Axiom 12 of the real numbers, $x = \sup A$ exists; we will show that $x^2 = p$ (and so x can be written as $\sqrt{p}$).

We will do this by showing $x^2 > p$ and $x^2 < p$ are both impossible, so by Axiom 10 $x^2 = p$ must be true.

Suppose $x^2 < p$. Let δ be the minimum of 1 and $\dfrac{p-x^2}{2x+1}$ (whichever is smaller). Then
both are positive

$$(x+\delta)^2 = x^2 + 2\delta x + \delta^2 \leq x^2 + 2\delta x + \delta$$

(here we used $\delta^2 \leq \delta$ since $\delta \leq 1$)

$$= x^2 + (2x+1)\delta$$

But δ is smaller than $\dfrac{p-x^2}{2x+1}$ as well, so

$$\leq x^2 + (\cancel{2x+1})\left(\frac{p-x^2}{(\cancel{2x+1})}\right) = x^2 + p - x^2 = p.$$

So $(x+\delta)^2 \leq p$, meaning $x+\delta \in A$. But this contradicts x being an upper bound for A. So it must be that $x^2 \geq p$.

Now suppose $x^2 > p$, and let $\delta = \dfrac{x^2-p}{2x}$, a positive quantity. Then

$$(x-\delta)^2 = x^2 - 2\delta x + \delta^2$$

$$\geq x^2 - 2\delta x \quad (\text{since } \delta^2 > 0)$$

$$= x - 2\left(\frac{x^2-p}{2x}\right)x = x^2 - x^2 + p = p.$$

But now $(x-\delta)^2 \geq p$ means that $x-\delta$ is an upper bound for A: Suppose $a \in A$ and $a > x-\delta$. Then $a^2 > (x-\delta)^2 \geq p$, contradicting $a \in A$. But $x-\delta$ being an upper bound contradicts $x = \sup S$. Thus we must have $x^2 = p$.

To finish the proof, we must show that the ~~same~~ theorem holds for p with $0<p<1$. But in this case $\frac{1}{p} > 1$, and the above argument shows there is a real number x with $x^2 = \frac{1}{p}$, so $\frac{1}{x^2} = p$. This proves the theorem for p between 0 and 1.

<u>Remark</u>: We finally know that an irrational number exists. We saw already that $\sqrt{2}$ is not rational, so if x with $x^2 = 2$ exists (this is what $\sqrt{2}$ means) it must be irrational. We now know it exists, thus $\sqrt{2} \in \mathbb{R} \setminus \mathbb{Q}$.

§0.5 Continued

We are proving how to derive some of the standard properties of the real numbers from the 12 axioms. We first prove:

Theorem: Let $x, y, z \in \mathbb{R}$. Then

(i) If $x<y$ then $-y<-x$

(ii) $0<1$

(iii) If $0<x<y$ then $0<\frac{1}{y}<\frac{1}{x}$

(iv) If $x<y$ and $z<0$, then $zy<zx$

(v) $0 \leq x^2$.

Proofs: (i). Suppose $x<y$.

Then $\quad x+(-x-y)<y+(-x-y)$ by Axiom 8.

We can remove the parantheses by Axiom 1

$$x+(-x)-y<y+(-x)-y$$

Reorder the terms by Axiom 2

$$x+(-x)-y<y+(-y)-x$$

Cancel x and $-x$, y and $-y$ by Axiom 5

$$0-y<0-x$$

Get rid of the zeroes by Axiom 4:

$$-y<-x.$$

This proves (i).

(ii) By axiom 10, either $0=1$, $0<1$ or $1<0$. We know $0 \neq 1$ (why?), so suppose $1<0$. Then $0<-1$, (by part (i)) and so $(-1)\cdot 0 < (-1)\cdot(-1)$. This gives $0<1$, a contradiction to the fact that we assumed $1<0$. Thus we must have $0<1$.

[Remark: The step $\underbrace{(-1)\cdot(-1)}_{=1}$ actually involves <u>many</u> steps if we reduce it to axioms – try this yourself]

(iii) Suppose $0<x<y$. By Axiom 10, $\frac{1}{x}=0$, $\frac{1}{x}<0$, or $0<\frac{1}{x}$. But $x\cdot\frac{1}{x}=1$, so we cannot have $\frac{1}{x}=0$. Suppose $\frac{1}{x}<0$. Then since $x>0$,

$$x\cdot\frac{1}{x} < x\cdot 0$$

$$\Rightarrow \quad \text{(by Axiom 11)}$$

$$\Rightarrow \quad 0 > 1, \text{ a contradiction.}$$

Thus $\frac{1}{x}>0$, same for $\frac{1}{y}$. Last, we calculate:

$0<x<y$ implies

$$0\cdot\frac{1}{x}\cdot\frac{1}{y} < x\cdot\frac{1}{x}\cdot\frac{1}{y} < y\cdot\frac{1}{x}\cdot\frac{1}{y} \quad \text{(by Axiom 11)}$$

$$\Rightarrow \quad 0 < \frac{1}{y} < \frac{1}{x}.$$

(iv) Suppose $x<y$ and $z<0$. Then $-z>0$, and by axiom 11 $-zx<-zy$, applying (i) gives $zy<zx$.

(v) If $x>0$, then by Axiom 11 $x \cdot x > x \cdot 0 = 0$. So $x^2 \geq 0$ in this case. If $x=0$ then $x^2=0$ so $x^2 \geq 0$ in this case. Last, if $x<0$ then $0 < -x$ by (i), so $0 < (-x)^2 = x^2$. Thus $0 \leq x^2$ in this case, too.

Axiom 12 says that least upper bounds exist. In fact, greatest lower bounds exist, too, but their existence can be derived from Axiom 12:

Theorem: If S is a nonempty set of real numbers that is bounded from below, then S has a greatest lower bound.

Proof: Let $T = \{t \mid t = -s \text{ for some } s \in S\}$. Now if M is a lower bound for S, then $M \leq s$ for all $s \in S$, therefore $-M \geq -s = t$ for all $t \in T$. Thus $-M$ is an upper bound for T. Similarly if K is an upper bound for T then $-K$ is a lower bound for S.

By Axiom 12, T has a least upper bound, call it B. Then $-B$ is a lower bound for S. Let C be any other lower bound, then $-C$ is an upper bound for T, thus $B \leq -C$ since B is the l.u.b of T. But then $C \leq -B$, so $-B$ is the greatest lower bound of S.

<u>Remark</u>: To show B is the g.l.b of a set S, show any other lower bound C satisfies $C \leq B$. (The "reverse" of a l.u.b).

Last, we prove something called the <u>Archimedean property</u>.

<u>Definition</u>: The real numbers are said to have the <u>Archimedean property</u> if, given any positive x and y in $\mathbb{R}$, there is an integer $n \in \mathbb{N}$ so that $nx > y$. (The real numbers do have this property).

First we prove something closely related:

<u>Theorem</u>: Let $x \in \mathbb{R}$ be given. Then there is an integer n such that $n \leq x < n+1$.

<u>Proof</u>: Let $A = \{n \mid n \in \mathbb{Z} \text{ and } n \leq x\}$. We will consider the cases of A empty and A nonempty.

If A is nonempty, then A is bounded from above (by x) so it has a least upper bound n_0. Therefore $n_0 - 1$ is not an upper bound for A, so there's $m \in A$ such that $n_0 - 1 < m \leq n_0$. But this means $n_0 < m+1$, so $m+1 \notin A$ since n_0 is an upper bound for A. But then $m+1 > x$, so overall, $m \leq x < m+1$.

On the other hand, suppose A is empty. Then $B = \{n \in \mathbb{Z} \mid n > x\} = \mathbb{Z}$. But B is nonempty and bounded from below, so B has a greatest lower bound z_0. But then $z_0 - 1$ is an integer not in B, contradicting $B = \mathbb{Z}$. Thus A cannot be empty.

Proof of Archimedean property from this theorem:

Let $x > 0$ be any positive number and ~~choose~~ let $y \in \mathbb{R}$ be given. Using the previous theorem, choose $n \in \mathbb{Z}$ such that $n - 1 \leq \frac{y}{x} < n$.

But then $\frac{y}{x} < n$ implies $y < nx$, since $x > 0$.

Thus the Archimedean property holds.

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Section 0.5 continued:

Having shown that $\sqrt{2} \in \mathbb{R}$ is irrational, we can show:

Theorem: The irrational numbers are dense in $\mathbb{R}$ (between any two real numbers there is an irrational number).

Proof: Suppose $x, y \in \mathbb{R}$ and $x < y$. Then because $\sqrt{2} > 0$, we have $\sqrt{2}x < \sqrt{2}y$. But now there is a rational number r with $\sqrt{2}x < r < \sqrt{2}y$, and we can assume that $r \neq 0$. (We can make this assumption because there are, in fact, infinitely many rationals between $\sqrt{2}x$ and $\sqrt{2}y$, so we can choose a nonzero one). But then

$$\sqrt{2}x < r < \sqrt{2}y$$

$$\Rightarrow x < \frac{r}{\sqrt{2}} < y.$$

Moreover, the number $\frac{r}{\sqrt{2}}$ is irrational: Suppose it were not, say $\frac{r}{\sqrt{2}} = \frac{p}{q}$ for some $p, q \in \mathbb{Z}$. Then $\sqrt{2} = \frac{rq}{p}$ (this step involves dividing by r, hence $r \neq 0$), meaning $\sqrt{2}$ is rational – a contradiction.

Remark: In fact (see the assignment): If p

is an integer that is not the square of another integer, then $\sqrt{p}$ is irrational.

Classifying real numbers as rational/irrational is just the beginning. Another way of classifying them is:

$$A = \text{Algebraic numbers} = \left\{ x \in \mathbb{R} \mid \begin{array}{l} \text{there is a polynomial} \\ p \text{ with } p(x) = 0 \end{array} \right\}$$

and $T = \mathbb{R} \setminus A$, the transcendental numbers.

It turns out A and T are also both dense subsets of $\mathbb{R}$, and A is countable (Exercise 37) while T is not. Note $\mathbb{Q} \subset A$, but A contains many more numbers — all square roots, for example.

Finally, we prepare ourselves in one last way for the study of limits. In future arguments, the absolute value function

$$|x| = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases}$$

will become extremely important, as the quantity $|x-y|$ can be thought of as the "distance from x to y". So when we want to study points being close to one another, or distances between points going to zero (as with limits) this function will be essential.

Theorem (properties of absolute value)

Let a and b be any real numbers. Then

(i) $|ab| = |a||b|$

(ii) If $\varepsilon > 0$, then $|a| < \varepsilon$ iff $-\varepsilon \leq a \leq \varepsilon$.

(iii) $|a+b| \leq |a| + |b|$ (triangle inequality)

(iv) $||a| - |b|| \leq |a-b|$

Proofs: It is useful to notice that $|x| = \sqrt{x^2}$ for all $x \in \mathbb{R}$, as then some proofs become easier.

(i) Given any two real numbers a and b,

$$|ab| = \sqrt{(ab)^2} = \sqrt{a^2b^2} = \sqrt{a^2}\sqrt{b^2} = |a||b|.$$

Of course this proof relies on the fact that $\sqrt{cd} = \sqrt{c} \cdot \sqrt{d}$ for any $c, d \in \mathbb{R}$ with $c, d > 0$, but proving this fact is easy. Note that $\sqrt{cd}$ and $\sqrt{c} \cdot \sqrt{d}$ are both solutions to $x^2 = cd$, since

$$(\sqrt{cd})^2 = cd = (\sqrt{c} \cdot \sqrt{d})^2.$$

Since there's a unique solution to $x^2 = cd$ with $x > 0$, $\sqrt{cd} = \sqrt{c} \cdot \sqrt{d}$.

can you see why?

(ii) Suppose $\varepsilon > 0$ and $-\varepsilon \leq a \leq \varepsilon$. If $a > 0$ then $|a| = a \leq \varepsilon$. If $a \leq 0$ then $|a| = -a$ and $-\varepsilon \leq a \Rightarrow -a \leq \varepsilon \Rightarrow |a| \leq \varepsilon$. So in either case ($a > 0$ or $a \leq 0$) we have $|a| \leq \varepsilon$.

Now if $|a| < \varepsilon$ and $a > 0$, then

$\cancel{-\varepsilon < 0 < a = |a| < \varepsilon, \text{ so } -\varepsilon \leq |a| < \varepsilon}$

$-\varepsilon < 0 < a = |a| < \varepsilon$, so $-\varepsilon < a < \varepsilon$.

On the other hand if $a \le 0$ then $|a| = -a$ and so $-a < \varepsilon$, hence $-\varepsilon < a \le 0 < \varepsilon$. Again $-\varepsilon < a < \varepsilon$, proving (ii).

(iii) Note that $|a| \ge a$ and $|b| \ge b$, so we immediately get $a+b \le |a| + |b|$. Also $-a \le |a|$ and $-b \le |b|$ so $-|a| \le a$ and $-|b| \le b$. Therefore $-|a| - |b| \le a+b$. Therefore we've arrived at $-|a|-|b| \le a+b \le |a|+|b|$, by part (ii) this happens if and only if $|a+b| \le |a| + |b|$.

(iv) By the triangle inequality, we can calculate

$$|a| = |a-b+b| \le |a-b| + |b|,$$ and so

$$|a| - |b| \le |a-b| + |b| - |b| = |a-b|.$$

Similarly we compute

$$\begin{aligned} |b| = |a-a+b| &\le |a| + |b-a| \\ &= |a| + |a-b| \end{aligned}$$

and therefore

$$|b| - |a| \le |a| + |a-b| - |a| = |a-b|.$$

So we have that both $|b|-|a|$ and $|a|-|b|$ are $\le |a-b|$. Since $||a|-|b||$ is equal to one of $|b|-|a|$ or $|a|-|b|$ (depending on which of $|a|$ or $|b|$ is larger), this proves $||a|-|b|| \le |a-b|$.

Thus (iv) is proved.

Note that we need to be adept at using these sorts of quantities and inequalities. For example, by (ii) we can argue that the following are equivalent:

(i) $|a-b| < \varepsilon$

(ii) $-\varepsilon < a-b < \varepsilon$

(iii) $a-\varepsilon < b < a+\varepsilon$

(iv) $b-\varepsilon < a < b+\varepsilon$.

In future proofs, we will use all of these statements interchangably with no reference to the fact that they are consequences of the previous theorem.

The above equivalence also explains why we view $|a-b|$ as a distance, for example:

$$X = \{x \mid |a-x| < \varepsilon\}$$
$$= \{x \mid a-\varepsilon < x < a+\varepsilon\}$$
$$= (a-\varepsilon, a+\varepsilon) \text{ by the equivalences above}$$

E.g.