MATH 2080

Lecture 2.

Sections 0.1 - 0.2

We can generalize De Morgan's Laws to arbitrary families of sets.

Suppose that Λ is a set, and for each $\lambda \in \Lambda$, we have specified a subset A_λ of a set S. Then the collection of all A_λ's is written $\{A_\lambda\}_{\lambda \in \Lambda}$, and

$$\bigcup_{\lambda \in \Lambda} A_\lambda = \{x \mid x \in A_\lambda \text{ for some } \lambda\}$$

$$\bigcap_{\lambda \in \Lambda} A_\lambda = \{x \mid x \in A_\lambda \text{ for all } \lambda \in \Lambda\}.$$

Example: Suppose that $B_n = (\frac{1}{n}, 1)$ for each $n \in \mathbb{N}$, and consider their union. We find

$\bigcup_{n \in \mathbb{N}} B_n = (0,1)$. To prove this:

Suppose $x \in (0,1)$. Then $1 > x > 0$ so there's an n such that $\frac{1}{n} < x$, meaning $x \in (\frac{1}{n}, 1) = B_n$. Thus $x \in \bigcup_{n \in \mathbb{N}} B_n$, and $(0,1) \subset \bigcup_{n \in \mathbb{N}} B_n$.

On the other hand if $x \in \bigcup_{n \in \mathbb{N}} B_n$ then $x \in (\frac{1}{n}, 1)$ for some n and so $x \in (0,1)$. Thus $\bigcup_{n \in \mathbb{N}} B_n \subseteq (0,1)$.

We conclude $\bigcup_{n \in \mathbb{N}} B_n = (0,1)$.

<u>Theorem</u> (De Morgan's Laws). Let $\{A_\lambda\}_{\lambda\in\Lambda}$ be a family of subsets of S. Then:

(i) $S\setminus\left(\bigcup_{\lambda\in\Lambda} A_\lambda\right)$

$$= \bigcap_{\lambda\in\Lambda}(S\setminus A_\lambda)$$

(ii) $S\setminus\left(\bigcap_{\lambda\in\Lambda} A_\lambda\right) = \bigcup_{\lambda\in\Lambda}(S\setminus A_\lambda)$

<u>Proof</u>: We'll do only (i).

Let $x\in S\setminus\left(\bigcup_{\lambda\in\Lambda} A_\lambda\right)$. Then $x\in S$ and $x\notin \bigcup_{\lambda\in\Lambda} A_\lambda$, so $x\in S$ and $x\notin A_\lambda$ for <u>any</u> λ. Thus $x\in S\setminus A_\lambda$ for every λ, so that $x\in\bigcap_{\lambda\in\Lambda}(S\setminus A_\lambda)$. We conclude that

$$S\setminus\left(\bigcup_{\lambda\in\Lambda} A_\lambda\right)\subset\bigcap_{\lambda\in\Lambda}(S\setminus A_\lambda).$$

On the other hand, let $x\in\bigcap_{\lambda\in\Lambda}(S\setminus A_\lambda)$. Then $x\in S\setminus A_\lambda$ for all λ, meaning $x\in S$ and x is never in A_λ for any λ. Thus x is not in $\bigcup_{\lambda\in\Lambda} A_\lambda$, so $x\in S\setminus\left(\bigcup_{\lambda\in\Lambda} A_\lambda\right)$.

We conclude $\bigcap_{\lambda\in\Lambda}(S\setminus A_\lambda)\subset S\setminus\left(\bigcup_{\lambda\in\Lambda} A_\lambda\right)$, proving (i).

<u>Example</u>: With $B_n = (\frac{1}{n}, 1)$:

$$\mathbb{R}\setminus\bigcup_{n\in\mathbb{N}} B_n = \mathbb{R}\setminus(0,1) = (-\infty,0]\cup[1,\infty)$$

$\|$ De Morgan's Laws

$\bigcap_{n\in\mathbb{N}}(\mathbb{R}\setminus B_n) \longleftarrow$ This is not so obvious without using De Morgan's Laws.

§0.2 Relations and functions.

Goal: Answer the question "What is a function?".

To do this we need the notion of a relation, which requires <u>Cartesian products</u>.

<u>Definition</u>: If A and B are sets, then the Cartesian product of A and B is

$$A \times B = \{(a,b) \mid a \in A \text{ and } b \in B\}.$$

I.e. it's the collection of all ordered pairs where the first element is from A, the second from B.

<u>Example</u>: If $A = \{0, 1, 2\}$ and $B = \{a, b, c\}$ then

$$A \times B = \{(0,a),(0,b),(0,c),(1,a),(1,b),(1,c),(2,a),(2,b),(2,c)\}.$$

<u>Remark</u>: Be careful! The notation (a,b) now has two meanings:

(a,b) – an ordered pair

$(a,b) = \{x \mid a < x < b\}$.

<u>Definition</u>: A <u>relation</u> from A to B is a subset $R \subset A \times B$. A function $f : A \longrightarrow B$ is a relation $f \subset A \times B$ such that if $(x,y) \in f$ ~~then~~ and $(x,z) \in f$ then $y = z$. Then

domain of $f = \{x \mid \text{there is a } y \text{ such that } (x,y) \in f\}$

image of $f = \{y \mid \text{there is an } x \text{ such that } (x,y) \in f\}$.

Whenever $(x,y) \in f$ we usually write $f(x) = y$.
If we write $f(x)=y$ instead of $(x,y) \in f$, then the domain and image of f become:

domain of $f = \{x \mid \text{there exists a } y \text{ such that } f(x)=y\}$

image of $f = \{y \mid \text{there exists an } x \text{ such that } f(x) = y\}$,

which are probably more familiar to you.

<u>Example</u>: Consider the function $f: \mathbb{Z} \longrightarrow \mathbb{Z}$ given by $f(n) = 2n$. Written as a relation,

$$f = \{(x,y) \in \mathbb{Z} \times \mathbb{Z} \mid y = 2x\}$$

i.e. it's the pairs $(n, 2n)$ for all $n \in \mathbb{Z}$. Here $\mathbb{Z}$ is the <u>codomain</u>, $2\mathbb{Z}$ is the image.

How do we talk about functions being 1-1, onto, or having inverses in this formal setup?

<u>Definition</u>: If $R \subset A \times B$ is a relation, then the converse of R, written $\hat{R}$, is

$$\hat{R} = \{(x,y) \in A \times B \mid (y,x) \in R\}.$$

(ie all the pairs in reverse order).

Recall that a function $f: A \longrightarrow B$ is 1-1 iff for all $x,y \in A$ $f(x) = f(y)$ implies $x = y$. This allows us to answer the question: When is the converse of a function $f \subset A \times B$ also a function?

Theorem: Suppose that $f \subset A \times B$ is a function. Then $\hat{f}$ is a function if and only if f is 1-1.

Proof: Assume $\hat{f}$ is a function, and suppose $f(x) = z$ and $f(y) = z$. Then $(x,z) \in f$ and $(y,z) \in f$, so $(z,x) \in \hat{f}$ and $(z,y) \in \hat{f}$. Since $\hat{f}$ is a function, this gives $x = y$ and so f is 1-1.

On the other hand, suppose f is 1-1 and that $(x,y) \in \hat{f}$ and $(x,z) \in \hat{f}$. Then $(y,x) \in f$ and $(z,x) \in f$, so $f(y) = x$ and $f(z) = x$. Thus $y = z$, since f is 1-1, meaning $\hat{f}$ is a function.

Thus when f is 1-1, there's a function $\hat{f}$ that goes with it. We will denote this function by f^{-1}, the inverse of f. I-e, our definition is:

$$f^{-1} = \{(x,y) \mid (y,x) \in f\}.$$

A bit more notation that you've probably seen before:

If $f: A \to B$ and $T \subset A$, then

$$f(T) = \{y \in B \mid \text{there's } x \in T \text{ with } f(x) = y\}$$
$$= \{f(x) \mid x \in T\}$$

and if $S \subset B$ then

$$f^{-1}(S) = \{a \in A \mid f(a) \in S\}.$$

Theses sets are called the image and inverse image respectively.

A function $f: A \longrightarrow B$ is called surjective or onto if $image(f) = B$.

MATH 2080

Lecture 3, Sections 0.2 - 0.3.

Last day we defined a function $f : A \to B$ as a subset $f \subset A \times B$, which is a relation with some additional special properties. We also introduced some standard terminology, like "inverse image" and "image" of a set.

We can compose such functions. If $f : A \to B$ and $g : B \to C$, then the composition of g with f is

$$g \circ f = \{(x,y) \in A \times C \mid \text{there is a } w \in B \text{ such that } (x,w) \in f \text{ and } (w,y) \in g\}.$$

and as usual, we write $g(f(x)) = y$ or $(g \circ f)(x) = y$

<u>Note, however</u>: Our definition of $g \circ f$ is clearly a relation (ie $g \circ f \subset A \times C$) but it is not necessarily a function! So we prove:

<u>Theorem</u>: If $f : A \to B$ and $g : B \to C$, then the definition of $g \circ f$ above yields a function $g \circ f : A \to C$.

<u>Proof</u>: Suppose that $(x,y) \in g \circ f$ and $(x, z) \in g \circ f$. We must show that $y = z$.

Since $(x,y) \in g \circ f$ there is a $w \in B$ such that $(x,w) \in f$ and $(w,y) \in g$. Similarly since $(x,z) \in g \circ f$ there's a $u \in B$ such that $(x,u) \in f$ and $(u,z) \in g$. But now $(x,w) \in f$ and $(x,u) \in f$ implies $w = u$ since f is a function. Then $(w,y) \in g$ and $(u,z) \in g$, combined with $w = u$ and the fact that g is a function, gives $z = y$. Thus $g \circ f$ is a function.

From this proof and definition, it should be clear that domain of $g \circ f$ is equal to domain of f.

<u>Example:</u> The fact that $\text{dom} f = \text{dom}\, g \circ f$ may disagree with what you learned in calculus, because our definitions are different.

E.g. If $f(x) = x + 1$ and $g(x) = \frac{1}{x}$, in calculus you might have seen:

Domain of f is $\mathbb{R}$

Domain of $g \circ f(x) = \frac{1}{x+1}$ is $\mathbb{R} \setminus \{-1\}$

But that is <u>not</u> how things work for us. In MATH 2080 our analysis would be:

$f(x) = x+1$ defines a function $f: \mathbb{R} \longrightarrow \mathbb{R}$

$g(x) = \frac{1}{x}$ defines a function $g: \mathbb{R}\setminus\{0\} \longrightarrow \mathbb{R}\setminus\{0\}$

Since the domain of g is not equal to the ~~image~~ codomain of f, we cannot compose them. So interpreted as above, $g \circ f$ is <u>not defined</u>. For $g \circ f$ to be defined, we must interpret the equation $f(x) = x+1$ to be a function $f: \mathbb{R}\setminus\{-1\} \longrightarrow \mathbb{R}\setminus\{0\}$, and then

<u>$\mathrm{dom}(f) = \mathrm{dom}(g \circ f)$.</u>

<u>§ 0.3</u> Mathematical induction.

Recall that induction is based on the well-ordering principle:

<u>Well-ordering principle</u>: Every nonempty subset of $\mathbb{N}$ has a smallest element.

Then the <u>principle of mathematical induction</u> is a theorem one can prove from this:

<u>Theorem</u> (Principle of induction).

Suppose $P(n)$ is a statement depending on a variable n which is either true or false for each n.

If

(i) $P(1)$ is true, and

(ii) for each $k \in \mathbb{N}$, $P(k)$ being true implies $P(k+1)$ is true then $P(n)$ is true for all $n \in \mathbb{N}$.

Proof: Suppose $P(n)$ is a ~~claim~~ statement as in the theorem, and that $P(n)$ is false for some n.

Set
$$S = \{ n \in \mathbb{N} \mid P(n) \text{ is false} \} \neq \emptyset.$$

By the well-ordering principle, S has a smallest element n_0. So $P(n_0)$ is false, and $n_0 > 1$ since $P(1)$ is true. Thus $P(n_0 - 1)$ is true.

But if $P(n_0 - 1)$ is true, then $P(n_0)$ is supposed to be true, by (ii). This is a contradiction, so $P(n)$ mustn't ever be false.

Example: Prove that
$$1 + x + x^2 + \dots + x^n = \frac{1 - x^{n+1}}{1 - x} \quad \text{if } x \neq 1 \text{ and } n \in \mathbb{N}.$$

Proof: Let $P(n)$ be the statement
$$1 + x + x^2 + \dots + x^n = \frac{1 - x^{n+1}}{1 - x} \quad \text{if } x \neq 1.$$

Then we observe:

(i): $P(1)$ is true, since $P(1)$ is
$$1 + x = \frac{1 - x^2}{1 - x} = \frac{(1-x)(1+x)}{1-x} = 1 + x \quad \text{if } x \neq 1.$$

Now we want to show

(ii) $P(n)$ true implies $P(n+1)$ true.

to do this, observe that if $P(n)$ is true then

$$1+x+x^2+\dots+x^n = \frac{1-x^{n+1}}{1-x} \quad \text{if } x \neq 1$$

and so

$$1+x+x^2+\dots+x^n+x^{n+1} = \frac{1-x^{n+1}}{1-x} + x^{n+1}$$

$$= \frac{1-x^{n+1}+x^{n+1}(1-x)}{1-x}$$

$$= \frac{1-x^{n+2}}{1-x} \quad \text{if } x \neq 1.$$

which is exactly $P(n+1)$. So $P(n)$ true $\Rightarrow P(n+1)$ true, proving that the formula holds for all n by induction

Remark: There is another trick for proving this formula that you should know.

Suppose $1+x+x^2+\dots+x^n = S$, and we try to solve for S. If $x \neq 1$ then

$$(1-x)(1+x+x^2+\dots+x^n) = S(1-x)$$

$$\Rightarrow 1+\cancel{x}+\cancel{x^2}+\dots+x^n-(\cancel{x}+\cancel{x^2}+\dots+x^{n+1}) = S(1-x)$$

cancel

$$1-x^{n+1} = S(1-x)$$

$$\Rightarrow S = \frac{1-x^{n+1}}{1-x}$$, which gives the formula we wanted.

There are strengthened forms of mathematical induction.

Theorem (Second principle of induction).
Suppose that $P(n)$ is a statement concerning a variable n. Then if

(i) $P(1), P(2), \ldots, P(m)$ are true, and

(ii) for $k \in \mathbb{N}$ if $P(i)$ is true for all i with $1 \le i \le k$, then $P(k+1)$ is true

then $P(n)$ is true for all n.

This principle is particularly useful when a function is defined recursively, e.g. the value of $f(n)$ depends in some way on the values of $f(1), f(2), \ldots, f(n-1)$.

Example: Define $f: \mathbb{N} \longrightarrow \mathbb{R}$ by

$f(1) = 3$, $f(2) = \frac{3}{2}$, and for $n \ge 3$ $f(n) = \frac{f(n-1) + f(n-2)}{2}$.

Claim: $f(n) = 2 + \left(-\frac{1}{2}\right)^{n-1}$.

Proof of claim: Let $P(n)$ be the ~~statement~~ that

$$f(n) = 2 + \left(-\tfrac{1}{2}\right)^{n-1}.$$

We will show $P(n)$ is true for all n using the previous theorem. First

(i) $P(1)$ is

$$f(1) = 2 + \left(-\tfrac{1}{2}\right)^{1-1} = 2 + 1 = 3$$

and $P(2)$ is

$$f(2) = 2 + \left(-\tfrac{1}{2}\right)^{2-1} = 2 - \tfrac{1}{2} = \tfrac{3}{2},$$

and bothe these claims are true. So (i) holds.

(ii) Assume we know $P(i)$ is true for i with $1 \le i \le k$. So $f(i) = 2 + \left(-\frac{1}{2}\right)^{i-1}$. Then

$$f(k+1) = \frac{f(k)+f(k-1)}{2} = \frac{2+\left(-\frac{1}{2}\right)^{k-1} + 2 + \left(-\frac{1}{2}\right)^{k-1-1}}{2}$$

$$= \frac{1}{2}\left(4 + \left(-\tfrac{1}{2}\right)^{k-2}\cdot\left(-\tfrac{1}{2}\right) + \left(-\tfrac{1}{2}\right)^{k-2}\right)$$

$$= \frac{1}{2}\left(4 + \tfrac{1}{2}\left(-\tfrac{1}{2}\right)^{k-2}\right)$$

$$= 2 + \left(-\tfrac{1}{2}\right)^{k},$$ so $P(k+1)$ is true.

MATH 2080

Section 0.3 Induction continued, + 0.4 Equiv. sets

We can modify both the principle of mathematical induction and the well-ordering principle to give further tools we can use in our proofs.

Modified well-ordering principle: If $S \subset \mathbb{Z}$, $S \neq \emptyset$, and S has a smallest element, then any nonempty subset of S has a smallest element.

This version follows from the original well-ordering principle as follows:

Proof: Suppose $S \subset \mathbb{Z}$, and s_0 is its smallest element. If $s_0 > 0$, then $S \subset \mathbb{N}$ and so every nonempty subset of S has a smallest element, by the "original" well-ordering principle.

On the other hand, suppose $s_0 < 1$. Then set $T = \{s \in S \mid s < 1\}$. The set T is finite, since it is contained in $\{s_0, s_0+1, \ldots, 0\}$. Now let $A \subset S$ be a nonempty subset. If $A \setminus T = A$, then $A \subset \mathbb{N}$ and so it has a smallest element. If $A \cap T \neq \emptyset$, then the smallest element of the finite set $A \cap T$ is the smallest element of A.

We have another method of induction, based on this:

<u>Theorem</u>: Suppose that $P(n)$ is a statement concerning a variable n. If

(i) For some $n_0 \in \mathbb{Z}$, $P(n_0)$ is true, and

(ii) for each $k \in \mathbb{Z}$ with $k \geq n_0$, if $P(k)$ is true then so is $P(k+1)$,

then $P(n)$ is true for all $n \in \mathbb{Z}$, $n \geq n_0$.

<u>Proof</u>: Left as exercise.

Here's how we would use this version of induction:

<u>Example</u>: Consider the statement $n + 10 < n^2$.

We test values of n:

$P(0)$: $0 + 10 < 0^2$ false.

$P(1)$: $1 + 10 < 1^2 = 1$, false

$P(2)$: $2 + 10 < 2^2 = 4$, false

$P(3)$: $3 + 10 < 3^2 = 9$, false

$P(4)$: $4 + 10 < 4^2 = 16$, true.

So maybe it's true for all $n \geq 4$? We attempt a proof:

Suppose $k \geq 4$, and ~~so~~ that $k + 10 < k^2$. Consider $k+1$. We find

$$(k+1) + 10 = (k+10) + 1 < k^2 + 1 < k^2 + 2k + 1 = (k+1)^2$$

our assumption $k+10 < k^2$ used here

adding $2k$ makes it bigger, since $k \geq 4$

Thus $(k+1)+10 < (k+1)^2$. So $P(k)$ true $\Rightarrow P(k+1)$ true, meaning $n+10 < n^2$ for all $n \geq 4$, by induction.

Section 0.4 Equivalent and countable sets

The goal of this section is to say when two sets have the same "size". For finite sets this is easy, you simply count elements in each set an compare results. What of infinite sets?

E.g: Which is "larger"?

- $\mathbb{N}$ • the set of all even numbers
- $\mathbb{Z}$ • $\mathbb{Q}$
- $\mathbb{R}$

Then the answer becomes tricky. We will answer these questions in the coming sections.

Definition: If A and B are sets, we say A is equivalent to B and write $A \sim B$ if and only if there's a 1-1 onto function $f: A \longrightarrow B$.

If we say "equivalence", we typically want this word to mean a few specific things, such as:

Theorem: Let A, B, C be sets. Then

(i) $A \sim A$ (reflexivity)

(ii) If $A \sim B$ then $B \sim A$ (symmetry)

(iii) If $A \sim B$ and $B \sim C$, then $A \sim C$ (transitivity)

Proof: If (i) is to be true, we need an onto 1-1 (bijective) function $A \longrightarrow A$. The function $f(a) = a$ (i.e "do nothing") works. This is called the identity function on A, and is denoted $1_A : A \to A$

(ii) To show that $A \sim B$ implies $B \sim A$, we start with a 1-1 onto function $f: A \longrightarrow B$. Since it's 1-1 and onto, this implies it has an inverse $f^{-1}: B \longrightarrow A$. The inverse function is also 1-1 and onto (we saw this already) and so $B \sim A$.

(iii) Assume that $A \sim B$ and $B \sim C$. Then there are 1-1 onto functions $f: A \to B$ and $g: B \longrightarrow C$. We need a 1-1 onto function from A to C, so we use $g \circ f: A \longrightarrow C$. We need to show $g \circ f$ is 1-1 and onto.

Lemma: If $f: A \longrightarrow B$ and $g: B \to C$ are both 1-1, so is $g \circ f$.

Proof: Suppose $g \circ f(a_1) = g \circ f(a_2)$. Then $g(f(a_1)) = g(f(a_2))$, so $f(a_1) = f(a_2)$ since g is 1-1. Then $a_1 = a_2$ since f is 1-1.

Lemma: If $f: A \longrightarrow B$ and $g: B \longrightarrow C$ are onto, so is $g \circ f$

Proof: Exercise