MATH 2080

§2.4 Continued

Last day, we discussed a function $f : [\alpha, \beta] \to \mathbb{R}$ and its "jumps" in terms of sets $\mathcal{U}(x)$ (upper) and $L(x)$ (lower).

Today we prove:

<u>Lemma</u>: Suppose $f : [\alpha, \beta] \longrightarrow \mathbb{R}$ is increasing.

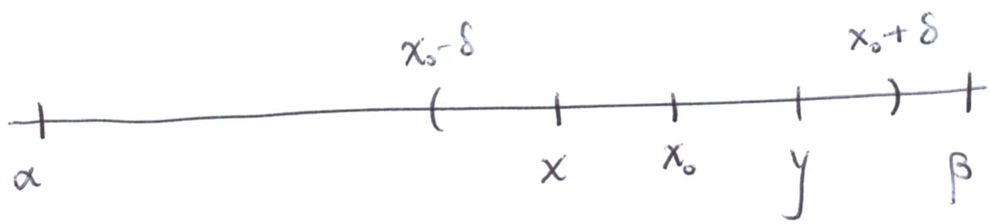
Set $\mathcal{U}(x) = \inf\{f(y) \mid x < y\}$ and $L(x) = \sup\{f(y) \mid x > y\}$ for all $x \in (\alpha, \beta)$. Then f has a limit at $x_0 \in (\alpha, \beta)$ if and only if $L(x_0) = \mathcal{U}(x_0)$, in which case

$$\lim_{x \to x_0} f(x) = f(x_0) = \mathcal{U}(x_0) = L(x_0).$$

<u>Proof</u>: First suppose that $\lim\limits_{x \to x_0} f(x) = A$, and we'll show $\mathcal{U}(x_0) = L(x_0) = A$.

Let $\varepsilon > 0$, and choose $\delta > 0$ such that $0 < |x - x_0| < \delta$ implies $|f(x) - A| < \varepsilon$ whenever $x \in [\alpha, \beta]$.

Now since $x_0 \in (\alpha, \beta)$ we can choose x, y such that



ie $\quad x_0 - \delta < x < x_0 < y < x_0 + \delta.$

Since f is increasing this gives

$$A - \varepsilon < f(x) \leq f(x_0) \leq f(y) < A + \varepsilon$$

By definition of $L(x_0)$, we know $L(x_0) \geq f(x)$ since it's the sup of $f(y)$, $y < x$. Similarly $U(x) \leq f(y)$, so

$$A - \varepsilon < f(x) \leq L(x_0) \leq f(x_0) \leq U(x_0) \leq f(y) < A + \varepsilon.$$

But this holds for any $\varepsilon > 0$, so we have $U(x_0) - L(x_0) < 2\varepsilon$ for all $\varepsilon > 0$. Thus $U(x_0) = L(x_0)$, and thus $U(x_0) = L(x_0) = f(x_0)$ from the inequalities above. Also from above, $A - \varepsilon < U(x_0) < A + \varepsilon$ for all $\varepsilon > 0$, so $A = U(x_0)$, ie.

$$\lim_{x \to x_0} f(x) = U(x_0) = L(x_0) = f(x_0).$$

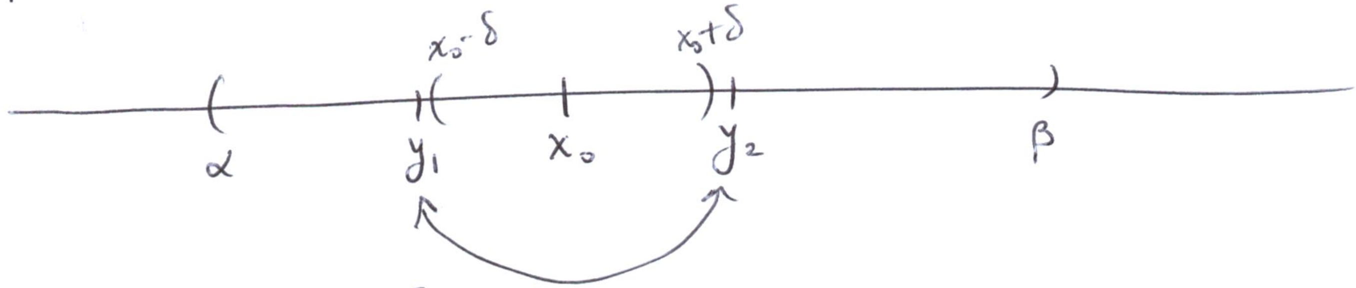
Conversely, suppose now that $U(x_0) = L(x_0)$. By the reasoning as above, we always have $U(x_0) \leq f(x_0) \leq L(x_0)$, so under our assumption $U(x_0) = f(x_0) = L(x_0)$. We must show $\lim_{x \to x_0} f(x) = f(x_0)$.

Let $\varepsilon > 0$.

Now $L(x_0) - \varepsilon$ is <u>not</u> an upperbound for $\{f(y) \mid y < x_0\}$, so there's a y_1 with

$$\alpha \leq y_1 < x_0 \text{ and } L(x_0) - \varepsilon < f(y_1).$$

Similarly there's y_2 with $x_0 < y_2 \leq \beta$ and $f(y_2) < U(x_0) + \varepsilon$. Set $\delta = \min\{x_0 - y_1, y_2 - x_0\}$.



These two points are witnesses to the fact that $U(x_0)+\varepsilon$ and $L(x_0)-\varepsilon$ are <u>not</u> lower and upper bounds of $\{f(y) \mid y > x_0\}$ and $\{f(y) \mid y < x_0\}$ respectively.

Now for $0 < |x_0 - x| < \delta$ we know $y_1 < x < y_2$, therefore

$$f(x_0) - \varepsilon = L(x_0) - \varepsilon < f(y_1) \leq f(x) \leq f(y_2) < U(x_0) + \varepsilon$$
$$= f(x_0) - \varepsilon,$$

since f is increasing. Thus $|f(x) - f(x_0)| < \varepsilon$, so f has a limit at x_0 and it's $f(x_0)$.

Now we have the most technical hurdle out of the way: For an increasing function $f : [\alpha, \beta] \to \mathbb{R}$, there's a limit at x_0 if and only if there's no jump there, ie. $U(x_0) = L(x_0)$.

Theorem: Suppose $f : [\alpha, \beta] \to \mathbb{R}$ is increasing. Then the set

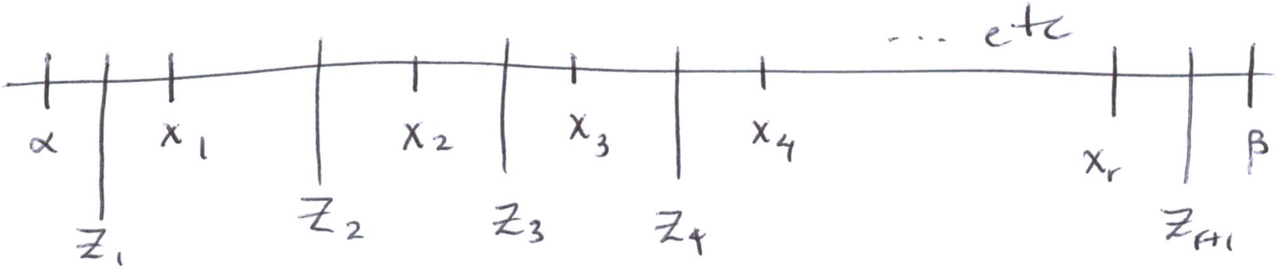
$$D = \{ x \in (\alpha, \beta) \mid \lim_{x \to x_0} f(x) \text{ does not exist} \}$$

is countable. If $x_0 \in (\alpha, \beta) \setminus D$, then $\lim_{x \to x_0} f(x) = f(x_0)$.

Proof: By our previous Lemma, $x \in D$ if and only if $U(x) \neq L(x)$. Since f is increasing, this happens if and only if $U(x) - L(x) > 0$. Set

$$J_n = \left\{ x \in (\alpha, \beta) \;\middle|\; U(x) - L(x) > \frac{1}{n} \right\}.$$

Then $D = \bigcup_{n \in \mathbb{N}} J_n$. The proof is finished if each J_n is finite. To see this, suppose $\{x_1, x_2, \dots, x_r\} \subset J_n$ and $\alpha < x_1 < x_2 < \dots < x_r < \beta$. Further choose z_i's so that they're between the x_i's:



Now for $i = 1, \dots, r$, $f(z_i) \le L(x_i)$ and $U(x_i) \le f(z_{i+1})$. Therefore $f(z_{i+1}) - f(z_i) \ge U(x_i) - L(x_i) > \frac{1}{n}$.

So we can rewrite $f(\beta) - f(\alpha)$ as a telescoping sum and get:

$$f(\beta) - f(\alpha) = f(\beta) - f(z_{i+1}) + \sum_{k=2}^{r+1} [f(z_k) - f(z_{k-1})] + f(z_1) - f(\alpha) \geq r\left(\frac{1}{n}\right)$$

$\leftarrow$ all terms $> \frac{1}{n}$.

Thus $(f(\beta) - f(\alpha))n > r$, so there can only be finitely many terms in J_n. Thus D is countable.

Last we observe: All this work applies to increasing functions only. However if f is decreasing, then $-f$ is increasing. So for decreasing functions we know $-f$ has only countably many ~~discont.~~ points where $\lim_{x \to x_0}(-f)$ does not exist, and $\lim_{x \to x_0}(-f) = -f(x_0)$. Multiplying everything by -1 proves the theorem holds for decreasing functions too.

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§3.1 Continuity

<u>Definition</u>: Suppose $E \subseteq \mathbb{R}$ and $f: E \to \mathbb{R}$. If $x_0 \in E$, then $f(x)$ is <u>continuous at</u> x_0 if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|x - x_0| < \delta$ and $x \in E$ implies $|f(x) - f(x_0)| < \varepsilon$.

Note that this is <u>almost</u> the definition of $\lim_{x \to x_0} f(x) = f(x_0)$, but not quite. If $x_0 \in E$ and x_0 is <u>not</u> an accumulation point of E, then our definition of $\lim_{x \to x_0} f(x)$ does not apply (it only applies at acc pts) while the above definition <u>does</u> apply.

If $x_0 \in E$ and x_0 is <u>not</u> an accumulation point, then choose $\delta > 0$ such that $(x_0 - \delta, x_0 + \delta) \cap E = x_0$. Then $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$ for any ε whatsoever, since only $x = x_0$ satisfies $|x - x_0| < \delta$.

In other words:

> If $x_0 \in E$ and x_0 is <u>not</u> an accumulation point of E, then f is continuous at x_0 "by default".

If x_0 IS an accumulation point, then:

Theorem: Let $f: E \longrightarrow \mathbb{R}$ with $x_0 \in E$ and x_0 an accumulation point of E. Then the following are equivalent:

(i) f is continuous at x_0

(ii) $\lim\limits_{x \to x_0} f(x) = f(x_0)$

(iii) For every sequence $\{x_n\}_{n=1}^{\infty}$ converging to x_0 with $x_n \in E$ for all n, $\{f(x_n)\}_{n=1}^{\infty}$ converges to $f(x_0)$.

Proof: We will show (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) $\Rightarrow$ (iii). Thus the truth of any one statement implies the truth of all others.

Assume (iii) holds. Then for the sequences $\{x_n\}_{n=1}^{\infty}$ with $x_n \in E \setminus \{x_0\}$ for all n, the sequence $\{f(x_n)\}_{n=1}^{\infty}$ converges to $f(x_0)$. By our previous work, we know this means $\lim\limits_{x \to x_0} f(x) = f(x_0)$. So that means (iii) $\Rightarrow$ (ii).

Now assume (ii). To show (i) holds, let $\varepsilon > 0$. Since (ii) holds, there's $\delta > 0$ such that

$0 < |x - x_0| < \delta$ and $x \in E$ implies $|f(x) - f(x_0)| < \varepsilon$.

This is almost the definition of continuity, except for $0 < |x - x_0|$ must be removed to get the definition of continuity. So suppose $x = x_0$. Then $|f(x) - f(x_0)| = 0 < \varepsilon$, so the definition of continuity is satisfied. So (ii) $\Rightarrow$ (i).

Finally suppose that (i) holds and we'll deduce (iii). To show (iii), let $\{x_n\}_{n=1}^{\infty}$ be a sequence converging to x_0 with $x_n \in E$. Let $\varepsilon > 0$. Then by continuity, there's $\delta > 0$ such that $|x - x_0| < \delta$ implies $|f(x) - f(x_0)| < \varepsilon$ whenever $x \in E$. By convergence of $\{x_n\}_{n=1}^{\infty}$ to x_0, there's an N such that $n \geq N$ implies $|x_n - x_0| < \delta$. So, for $n \geq N$ we get $|f(x_n) - f(x_0)| < \varepsilon$. But this exactly means that $\{f(x_n)\}_{n=1}^{\infty}$ converges to $f(x_0)$.

Example: Recall the function

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \\ \frac{1}{q} & \text{if } x = \frac{p}{q} \in \mathbb{Q} \text{ with } \frac{p}{q} \text{ in lowest terms.} \end{cases}$$

We saw that $\lim_{x \to x_0} f(x) = 0$ at every x_0, so in particular, when $f(x_0) = 0$ (ie at x_0 irrational) we have $\lim_{x \to x_0} f(x) = f(x_0) = 0$. When x_0 is rational, say $x_0 = \frac{p}{q}$, then $\lim_{x \to x_0} f(x) \neq \frac{1}{q}$, so we conclude:

The function $f(x)$ is continuous at every point of $\mathbb{R} \setminus \mathbb{Q}$ and discontinuous elsewhere.

Example: Consider the function $f(x) = \dfrac{x-1}{x^2-1}$.

This function is equal to $\dfrac{1}{x+1}$ at all points except $x=1$, where there is a hole. We saw that limits do not detect the difference between $f(x)$ and $\dfrac{1}{x+1}$ since $\lim\limits_{x \to x_0} f(x) = \lim\limits_{x \to x_0} \dfrac{1}{x+1}$ no matter the value of x_0. However, there IS a difference when it comes to continuity:

$\lim\limits_{x \to \cancel{x_0}} f(x) = \frac{1}{2} \neq f(1)$, since $f(1)$ is not defined.

So f is not continuous at 1. On the other hand,

$\lim\limits_{x \to 1} \dfrac{1}{x+1} = \dfrac{1}{1+1} = \dfrac{1}{2}$, so $\dfrac{1}{x+1}$ IS continuous at $x=1$.

Example: If $g(x) = \sin\left(\frac{1}{x}\right)$ where $0 < x < 1$, and $g(0) = 38$, then $\lim\limits_{x \to 0} g(x)$ does not exist (and 0 is an accumulation point of the domain) so $g(x)$ is not continuous at 0.

On the other hand if $g(x) = \sin\left(\frac{1}{x}\right)$ for $\frac{1}{100} < x < 1$ and $g(0) = 38$, then g IS continuous at $x=0$ since 0 is not an accumulation point of the domain of g (domain $= (\frac{-1}{100}, 1) \cup \{0\}$).

Added material (not in book).

We can also define continuity of functions $\mathbb{R}^2 \to \mathbb{R}$ or $\mathbb{R}^2 \to \mathbb{R}^2$ or $\mathbb{R} \to \mathbb{R}^2$ as well, $\mathbb{R}^n \to \mathbb{R}^m$, etc.

We cover functions of two variables.

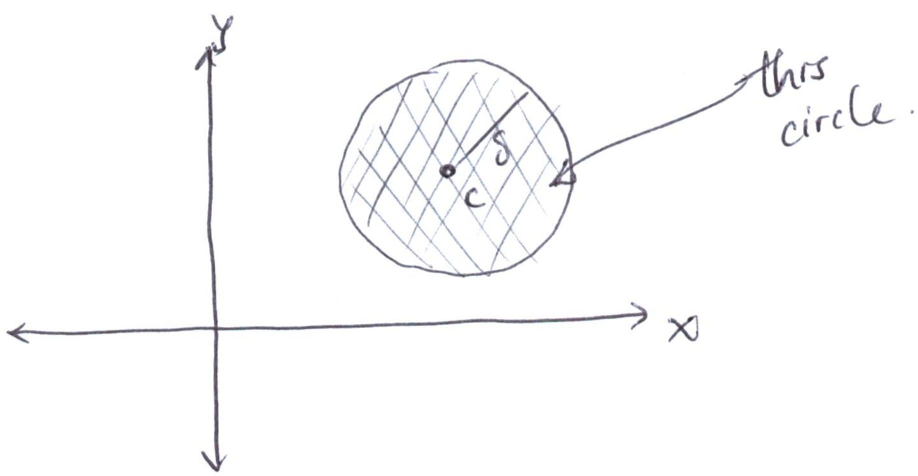
Definition: Let $A \subseteq \mathbb{R}^2$. A function $f: \mathbb{R}^2 \to \mathbb{R}$ is continuous at $c \in A$ if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\| x - c \| < \delta \text{ implies } |f(x) - f(c)| < \varepsilon.$$

Note: Here $\| \cdot \|$ is the Euclidean distance/norm, meaning if $x = (x_1, x_2)$ and $c = (c_1, c_2)$ then

$$\| x - c \| = \| (x_1 - c_1, x_2 - c_2) \| = \sqrt{(x_1 - c_1)^2 + (x_2 - c_2)^2}.$$

i.e. the set $\{ x \in \mathbb{R}^2 \mid \| x - c \| < \delta \}$ is



We have theorems identical to our previous ones:

<u>Theorem</u>: Let $A \subseteq \mathbb{R}^2$, $f: A \to \mathbb{R}$ a function and $c \in A$ an accumulation point of A. Then the following are equivalent:

(i) f is continuous at c.

(ii) Suppose $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are sequences in $\mathbb{R}$ such that $(x_n, y_n) \in A$, and if $c = (c_1, c_2)$ then $x_n \neq c_1$ for all n and $y_n \neq c_2$ for all n. If $\{x_n\}_{n=1}^{\infty}$ converges to c_1 and $\{y_n\}_{n=1}^{\infty}$ converges to c_2 then $\{f(x_n, y_n)\}_{n=1}^{\infty}$ converges to $f(c)$.