MATH 2080

Section 2.3 Algebra of limits

As with convergence of sequences of the forms $\{a_n + b_n\}_{n=1}^{\infty}$, $\{a_n b_n\}_{n=1}^{\infty}$, $\left\{\frac{a_n}{b_n}\right\}_{n=1}^{\infty}$, these operations also behave predictably with respect to limits of functions. Recall that if $f, g : D \to \mathbb{R}$ then

$(f \pm g)(x) : D \to \mathbb{R}$ is $(f \pm g)(x) = f(x) \pm g(x)$

$(fg)(x) : D \to \mathbb{R}$ is $(fg)(x) = f(x)g(x)$

$\frac{f}{g} : D \to \mathbb{R}$ is $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$

Theorem: Suppose $f, g : D \to \mathbb{R}$ and x_0 is an accumulation point of D. Suppose $\lim_{x \to x_0} f(x)$ and $\lim_{x \to x_0} g(x)$ exist. Then:

① $f+g$ has a limit at x_0, and

$$\lim_{x \to x_0} (f+g)(x) = \lim_{x \to x_0} f(x) + \lim_{x \to x_0} g(x)$$

② fg has a limit at x_0, and

$$\lim_{x \to x_0} (fg)(x) = \left(\lim_{x \to x_0} f(x)\right)\left(\lim_{x \to x_0} g(x)\right)$$

③ If $g(x) \neq 0$ for all $x \in D$ and $\lim_{x \to x_0} g(x) \neq 0$, then $\left(\frac{f}{g}\right)(x)$ has a limit at x_0 and

$$\lim_{x \to x_0} \left(\frac{f}{g}\right)(x) = \frac{\lim_{x \to x_0} f(x)}{\lim_{x \to x_0} g(x)}.$$

Proof: ① We could prove this using sequences as follows: If $\{x_n\}_{n=1}^{\infty}$ is any sequence converging to x_0 with $x_n \in D$ for all n and $x_0 \neq x_n$ for all n. Then we need only show that $\{(f+g)(x_n)\}_{n=1}^{\infty}$ converges to $\lim_{x\to x_0} f(x) + \lim_{x\to x_0} g(x)$. By assumption, $\lim_{x\to x_0} f(x)$ and $\lim_{x \to x_0} g(x)$ exist, so $\{f(x_n)\}_{n=1}^{\infty}$ and $\{g(x_n)\}_{n=1}^{\infty}$ converge to these limits respectively. Since a sum of sequences converges to a sum of limits, $\{(f+g)(x_n)\}_{n=1}^{\infty}$ converges to the same thing as $\{f(x_n)+g(x_n)\}_{n=1}^{\infty}$ which converges to $\lim_{x\to x_0} f(x) + \lim_{x\to x_0} g(x)$. This proves the claim.

An alternative proof goes as follows: Suppose $\lim_{x\to x_0} f(x) = L$ and $\lim_{x\to x_0} g(x) = M$; and let $\varepsilon > 0$. Choose δ' and δ'' such that $0 < |x - x_0| < \delta'$ implies $|f(x) - L| < \varepsilon/2$ and $0 < |x - x_0| < \delta''$ implies $|g(x) - M| < \varepsilon/2$. Set $\delta = \min\{\delta', \delta''\}$. Then for $0 < |x - x_0| < \delta$, we compute

$$\begin{aligned} |(f+g)(x) - (M+L)| &= |f(x) - L + g(x) - M| \\ &\leq |f(x) - L| + |g(x) - M| \\ &< \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

(2) Set $A = \lim_{x \to x_0} f(x)$ and $B = \lim_{x \to x_0} g(x)$. Let $\varepsilon > 0$.

We need $\delta > 0$ such that $0 < |x - x_0| < \delta$ implies $|(fg)(x) - AB| = |f(x)g(x) - AB| < \varepsilon$. Last day we saw that there exists $\delta_1 > 0$ and $M > 0$ such that $0 < |x - x_0| < \delta_1$ and $x \in D$ implies $|f(x)| \leq M$. Set

$$\varepsilon' = \frac{\varepsilon}{|B| + M} > 0.$$

Now since $\lim_{x \to x_0} f(x) = A$ and $\lim_{x \to x_0} g(x) = B$, we can choose $\delta_2 > 0$ such that $0 < |x - x_0| < \delta_2$ and $x \in D$ implies $|f(x) - A| < \varepsilon'$, and $\delta_3 > 0$ such that $0 < |x - x_0| < \delta_3$ and $x \in D$ implies $|g(x) - B| < \varepsilon'$. Set $\delta = \min\{\delta_1, \delta_2, \delta_3\}$ so that all inequalities above hold when $0 < |x - x_0| < \delta$ and $x \in D$. Then for such x, we calculate:

$$\begin{aligned}
|(fg)(x) - AB| &= |f(x)g(x) - AB| \\
&\leq |f(x)g(x) - f(x)B| + |f(x)B - AB| \\
&= |f(x)|\,|g(x) - B| + |B|\,|f(x) - A| \\
&< M\varepsilon' + |B|\varepsilon' \\
&= \frac{\varepsilon}{|B| + M}(M + |B|) = \varepsilon.
\end{aligned}$$

<u>Remark</u>: Return to the analagous proof for sequences and compare!

③ Again, as in ① this can be proved using sequences or directly. We use sequences.

Suppose $\{x_n\}_{n=1}^{\infty}$ is a sequence converging to x_0 and that $x_n \in D$, $x_n \neq x_0$ for all n. Then by our assumptions (and a theorem from last week) $\{f(x_n)\}_{n=1}^{\infty}$ and $\{g(x_n)\}_{n=1}^{\infty}$ converge to $\lim_{x \to x_0} f(x)$ and $\lim_{x \to x_0} g(x)$ respectively. Since $g(x) \neq 0$ for all $x \in D$, we know $g(x_n) \neq 0$ $\forall$ x_n, and by assumption we also know $\lim_{x \to x_0} g(x) \neq 0$, so $\{g(x_n)\}_{n=1}^{\infty}$ converges to something nonzero. Thus

$$\left\{ \left(\frac{f}{g}\right)(x_n) \right\}_{n=1}^{\infty} = \left\{ \frac{f(x_n)}{g(x_n)} \right\}_{n=1}^{\infty} \text{ converges to } \frac{\lim_{x \to x_0} f(x)}{\lim_{x \to x_0} g(x)},$$

and so $\lim_{x \to x_0} \left(\frac{f}{g}\right)(x) = \dfrac{\lim_{x \to x_0} f(x)}{\lim_{x \to x_0} g(x)}$ as required.

As with sequences, we can compare limits if the functions can be compared:

<u>Theorem</u>: Suppose $f, g : D \to \mathbb{R}$ and x_0 is an accumulation point of D. If $\lim_{x \to x_0} f(x)$ and $\lim_{x \to x_0} g(x)$ exist and $f(x) \leq g(x)$ for all $x \in D$, then

$$\lim_{x \to x_0} f(x) \leq \lim_{x \to x_0} g(x).$$

Proof: Exercise, it can be done using sequences or directly.

Example: Consider $f:(0,1) \to \mathbb{R}$ defined by $f(x) = x\sin(\frac{1}{x})$.
In MATH 1500, you could show $\lim_{x\to 0} x\sin(\frac{1}{x}) = 0$ using the squeeze theorem. We can argue directly by observing that $-1 \le \sin(x) \le 1$, so

$$|f(x)| = |x\sin(\tfrac{1}{x})| \le |x|,$$

therefore if $0 < |x| < \delta$ then with $\delta = \varepsilon$ we get

$$|f(x) - 0| = |f(x)| \le |x| < \delta = \varepsilon. \text{ So } \lim_{x\to 0} f(x) = 0.$$

In fact in this example there is nothing special about $\sin(\frac{1}{x})$ aside from being bounded, and nothing special about x aside from $\lim_{x\to x_0} x = 0$. This suggests a theorem:

Theorem: Suppose $f, g: D \to \mathbb{R}$ and x_0 is an accumulation point of D. Suppose f is bounded in a neighbourhood of x_0 and $\lim_{x\to x_0} g(x) = 0$. Then $\lim_{x\to x_0} (fg)(x) = 0$.

Proof: Let $\varepsilon > 0$. Then there is $\delta_1 > 0$ and $M > 0$ such that $|f(x)| \le M$ whenever $x \in D$ and $|x - x_0| < \delta_1$.

Set $\varepsilon' = \frac{\varepsilon}{M}$. Then there exists δ_2 such that if $x \in D$ and $0 < |x - x_0| < \delta_2$ then $|g(x) - 0|$

$$= |g(x)| < \varepsilon'.$$

Choose $\delta = \min\{\delta_1, \delta_2\}$. Then $0 < |x - x_0| < \delta$ and $x \in D$ implies

$$|(fg)(x)| = |f(x)g(x)| = |f(x)|\,|g(x)| \leq M\varepsilon' = \varepsilon.$$

So fg has the required limit at $x = x_0$.

Section 2.3 continued.

Using the previous theorem we can handle a large class of functions. First note:

Example: If $f(x) = x$, then $\lim_{x \to x_0} f(x) = x_0$, because if $\varepsilon > 0$ then $\delta = \varepsilon$ gives $0 < |x - x| < \delta$ implies $|f(x) - x_0| = |x - x_0| < \delta = \varepsilon$. Similarly easy is: If $c \in \mathbb{R}$ and $g(x) = c$ for all $x \in \mathbb{R}$, then $\lim_{x \to x_0} g(x) = c$.

Now we can prove:

- Since x^n is a product of x with itself n times, and since the limit of a product is the product of the limits,
$$\lim_{x \to x_0} x^n = x_0^n$$
- Since $c x^n$ is the product of functions $g(x) = c$ and $f(x) = x^n$, the limit is
$$\lim_{x \to x_0} c x^n = \lim_{x \to x_0} c \cdot \lim_{x \to x_0} x^n = c x_0^n$$
- If $p(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$, where $a_i \in \mathbb{R}$, then as the limit of a sum is the sum of the limits we get

$$\lim_{x\to x_0}(a_0+a_1x+\dots+a_nx^n)=\sum_{i=0}^{n}\left(\lim_{x\to x_0}a_ix^i\right)$$

$$=a_0+a_1x_0+\dots+a_nx_0^n=p(x_0).$$

- If $p(x)$ and $q(x)$ are polynomials, and $\{r_1,\dots,r_n\}$ are the roots of $q(x)$ (ie $q(r_i)=0$ for each $i=1,\dots,n$), then

$$\lim_{x\to x_0}\frac{p(x)}{q(x)}=\frac{p(x_0)}{q(x_0)}$$ provided that x_0 is not equal to r_i for some i. This follows because under the condition that q has roots at only $\{r_1,\dots,r_n\}$ (not at x_0!) we can find a neighbourhood of x_0 where q is nonzero. Then our theorem that states $$\lim_{x\to x_0}\left(\frac{f}{g}\right)(x)=\frac{\lim_{x\to x_0}f(x)}{\lim_{x\to x_0}g(x)}$$ applies on that neighbourhood. At $x_0=r_i$, the limit is more subtle and we must deal with this later in the course. We can also prove:

<u>Theorem</u>: Suppose $f:D\to\mathbb{R}$ with x_0 an accumulation point of D. If $\lim_{x\to x_0}f(x)=L$, then

$$\lim_{x\to x_0}\sqrt{f(x)}=\sqrt{L},$$

provided $f(x)\geq 0$ for all x in $D\cap Q$, where

Q is a neighbourhood of x_0.

<u>Proof</u>: We use the fact that if $\{a_n\}_{n=1}^{\infty}$ converges to L, then $\{\sqrt{a_n}\}_{n=1}^{\infty}$ converges to $\sqrt{L}$, and mimic the other sequence/limit proofs.

<u>Example</u>: We can now do most "MATH 1500" limits in a rigourous way. For example, if $h : (0,1) \to \mathbb{R}$ has formula $h(x) = \dfrac{\sqrt{4+x} - 2}{x}$

then we can calculate $\lim_{x \to 0} h(x)$ via:

$$\lim_{x\to 0} \frac{\sqrt{4+x} - 2}{x} = \lim_{x\to 0} \frac{\sqrt{4+x} - 2}{x} \cdot \frac{\sqrt{4+x} + 2}{\sqrt{4+x} + 2}$$

$$= \lim_{x\to 0} \frac{\cancel{x}}{\cancel{x}(\sqrt{4+x} + 2)}$$

Now the denominator is a function which, by our previous remarks and theorems, has limit $\lim_{x\to 0} \sqrt{4+x} + 2 = \sqrt{0+4} + 2 = 4$.

This is nonzero, so

$$\lim_{x\to 0} \frac{1}{\sqrt{4+x} + 2} = \frac{\lim_{x\to 0} 1}{\lim_{x\to 0} \sqrt{4+x} + 2}$$

$$= \frac{1}{4}.$$

§2.4 Limits of monotone functions.

Not surprisingly, just as monotone sequences exhibited special behaviour with respect to convergence, so do monotone functions with respect to limits.

Definition: Let $f: D \to \mathbb{R}$. A function f is

- increasing if for all $x, y \in D$ with $x \le y$ we have $f(x) \le f(y)$
- decreasing if for all $x, y \in D$ with $x \le y$ we have $f(x) \ge f(y)$.

A function which is either increasing or decreasing is monotone.

For sequences, the result was: monotone bounded sequences have a limit.

For functions, will the result be similar? Do monotone bounded functions always have a limit at some point? At every point?

Example: If $f(x) = [x]$, the greatest integer function, then $f(x)$ is increasing. However $\lim_{x \to x_0} f(x)$ does not exist whenever $x_0 \in \mathbb{Z}$. So clearly $f(x)$ is not required to have a limit at every x_0.

What if we bound $f(x)$? Still no, because we could just use $f:[0,2]\to\mathbb{R}$, $f(x)=[x]$ to produce a bounded increasing function with a problem at $x_0 = 1$.

It turns out that f monotone implies that $\lim_{x\to x_0} f(x)$ can only fail to exist in a particular way: There must be a "jump". Specifically, if $f:[\alpha,\beta]\to\mathbb{R}$ and $\alpha < x < \beta$, set

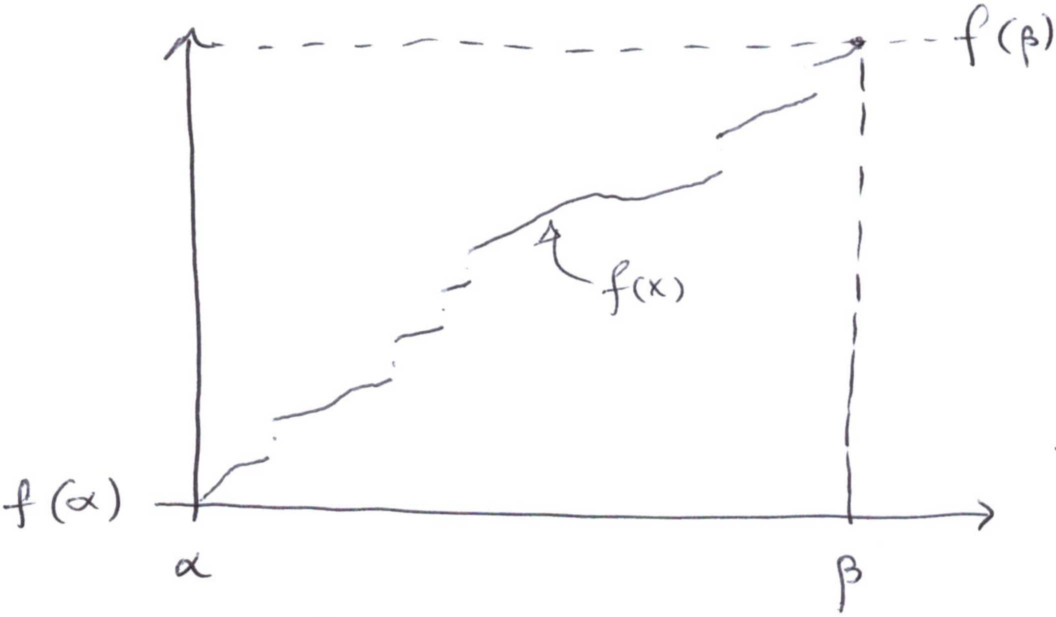
$$U(x) = \inf\{f(y) \mid x<y\} \quad \text{and}$$
$$L(x) = \sup\{f(y) \mid y<x\}.$$

Then $f(\alpha) \le f(x) \le f(\beta)$ for all $x\in[\alpha,\beta]$ when f is increasing, $U(x)$ and $L(x)$ are always defined. Now $U(x)-L(x)$ measures the size of the "jump" at x, and it will turn out that $\lim_{x\to x_0} f(x)$ exists if and only if $U(x_0)-L(x_0)=0$. Then set

$$J_n = \{x\in(\alpha,\beta) \mid U(x)-L(x) > \tfrac{1}{n}\}$$

ie. J_n = all x's in (α,β) where $f(x)$ jumps by more than $\frac{1}{n}$.

Each J_n will be finite, since the sum of all the jumps should be less than $f(\beta)-f(\alpha)$ since f is increasing:



6 vertical jumps by amounts bigger than $1/n$ cannot add up to more than $f(\beta) - f(\alpha)$.

Thus, $\bigcup_{n=1}^{\infty} J_n$, the set of all points where $f(x)$ has a jump, will be countable. Thus we suspect:

<u>Theorem</u>: Suppose $f : [\alpha, \beta] \longrightarrow \mathbb{R}$ is monotone. Then the set $D = \{ x_0 \in (\alpha, \beta) \mid \lim_{x \to x_0} f(x) \text{ does not exist} \}$ is countable. Moreover if $\lim_{x \to x_0} f(x)$ exists (ie if $x_0 \notin D$) then $\lim_{x \to x_0} f(x) = f(x_0)$.

We will prove this next day.