MATH 2080 Test 2

Q1

a) Every bounded infinite subset of the real numbers has at least one accumulation point.

b) For every $\varepsilon > 0$ there exists $\delta > 0$ such that $0 < |x - x_0| < \delta$ implies $|f(x) - L| < \varepsilon$.

c) For every $L \in \mathbb{R}$ there exists $\varepsilon > 0$ such that for all $\delta > 0$ there exists x with $0 < |x - x_0| < \delta$ such that $|f(x) - L| \geq \varepsilon$.

d) Let $\{a_n\}_{n=1}^{\infty}$ be a sequence, and $\{n_k\}_{k=1}^{\infty}$ a sequence of natural numbers with $n_1 < n_2 < n_3 < \dots$. Then $\{a_{n_k}\}_{k=1}^{\infty}$ is a subsequence of $\{a_n\}_{n=1}^{\infty}$.

e) A point $x_0 \in \mathbb{R}$ is an accumulation point of $S \subseteq \mathbb{R}$ if every neighbourhood of x_0 contains a point of S other than x_0.

f) Consider $a_n = 1$ for all n. Then $\{a_n\}_{n=1}^{\infty}$ converges to 1 but the set $\{a_n \mid n \in \mathbb{N}\}$ has no accumulation points because it is finite.

Q2 Let $\varepsilon > 0$. We need to find δ such that $0 < |x-1| < \delta$ implies $|f(x) - 4| < \varepsilon$.

We compute $|(x^2+3) - 4| < \varepsilon$

$$\Leftrightarrow |x^2 - 1| < \varepsilon$$

$$\Leftrightarrow |x-1||x+1| < \varepsilon.$$

Now as long as $\delta \leq 1$, we know $|x+1|$ will take as inputs $x \in [0,2]$ and so $|x+1| \leq 3$.

So if $\delta \leq \frac{\varepsilon}{3}$ and $\delta \leq 1$ then

$$|(x^2+3) - 4| = |x-1||x+1| < \delta \cdot \frac{\varepsilon}{3} = \varepsilon.$$

Q3

Suppose $\{a_n\}_{n=1}^{\infty}$ is increasing and bounded. Then $\{a_n \mid n \in \mathbb{N}\}$ is bounded above, so $s = \sup\{a_n \mid n \in \mathbb{N}\}$ exists.

Let $\varepsilon > 0$. Suppose $a_n \leq s - \varepsilon$ for all n. Then $s-\varepsilon$ would be an upper bound for $\{a_n \mid n \in \mathbb{N}\}$ meaning s is not the sup, a contradiction. So, $\exists n$ such that $a_n > s - \varepsilon$. But then since the sequence is increasing, $s \geq a_m \geq a_n > s - \varepsilon$ for all $m \geq n$, and thus $|a_m - s| < \varepsilon$ for all $m \geq n$. Therefore $\{a_n\}_{n=1}^{\infty}$ converges to s.

Q4 Suppose $f : \mathbb{R} \to \mathbb{R}$ is one-to-one on $(x_0 - \delta, x_0 + \delta)$, and $\lim_{x \to x_0} f(x) = L$. Let Q be a neighbourhood of L, and choose ε such that $(L-\varepsilon, L+\varepsilon) \subset Q$. Choose $\delta' < \delta$ such that $0 < |x - x_0| < \delta'$ implies $|f(x) - L| < \varepsilon$.

Since x_0 is an accumulation point of S, there's $y \in S$ with $|y - x_0| < \delta'$. By choosing a different y if necessary, we can ensure that $f(y) \neq L$ since f is one-to-one. Then $f(y) \in (L-\varepsilon, L+\varepsilon)$ shows that L is an accumulation point of $f(S)$.

The claim fails if $f(x) = c$ is constant, since $f(S)$ is finite and so has no accumulation points no matter the set S.

Alternative solution:

Choose a ~~such~~ sequence $\{x_n\}_{n=1}^{\infty}$ of points in S, $x_n \neq x_0$ for all n, converging to x_0. We can assume $\{x_n\}_{n=1}^{\infty}$ is not eventually constant since x_0 is an accumulation point of S. Since $\lim_{x \to x_0} f(x) = L$, $\{f(x_n)\}_{n=1}^{\infty}$ converges to L. Moreover, if f is one-to-one on $(x_0 - \delta, x_0 + \delta)$ then all points in $\{x_n \mid n \in \mathbb{N}\} \cap (x_0 - \delta, x_0 + \delta]$ are mapped to distinct points $f(x_n)$, so $\{f(x_n)\}_{n=1}^{\infty}$ is not eventually constant. Thus L is an accumulation point of $\{f(x_n) \mid n \in \mathbb{N}\}$, and therefore of $f(S)$.