Q1 a) (i) A permutation of a set X is a bijection
$\sigma : X \to X$.

(ii) A cycle is a permutation σ satisfying:
There exist $a_1, \ldots, a_k$ such that

$$\sigma(a_1) = a_2,\ \sigma(a_2) = a_3,\ \ldots,\ \sigma(a_k) = a_1$$

and $\sigma(x) = x$ for all $x \notin \{a_1, \ldots, a_k\}$.

(iii) A transposition is a cycle of length 2.

b) An even permutation is a permutation that can be written as an ~~even~~ product of an even number of transpositions.

c) Consider the elements $(1,2)(3,4) \in A_n$ for $n \geq 4$ and $(2,3)(1,3) \in A_n$ $(n \geq 4)$. We compute:

$$(2,3)(1,3)(1,2)(3,4) = (1,3,4) \quad \text{and}$$

$$(1,2)(3,4)(2,3)(1,3) = (2,3,4),$$ they do not commute.

So that A_n, $n \geq 4$, is not abelian, since $A_4 \subseteq A_n$ for all $n \geq 4$.

Q2: a) If $H \leq G$ is a subgroup, then $\forall g \in G$
$gH = \{gh \mid h \in H\}$ is a left coset, and
$Hg = \{hg \mid h \in H\}$ is a right coset.

b) The cosets of D_n all have size $|D_n| = 2n$,
and they partition S_n. Therefore there are
$|S_n| / |D_n| = \frac{n!}{2n}$ cosets.

c) A subgroup $N \subset G$ is <u>normal</u> in G if $gN = Ng$
for all $g \in G$.

d) Suppose that N_1 and N_2 are normal. Then
$gN_i g^{-1} = N_i$ for $i = 1, 2$. Therefore

$$g(N_1 \cap N_2)g^{-1} = gN_1 g^{-1} \cap gN_2 g^{-1} = N_1 \cap N_2$$

So that $N_1 \cap N_2$ is normal. One can check
that $g(N_1 \cap N_2)g^{-1} = gN_1 g^{-1} \cap gN_2 g^{-1}$ by writing them in full:

$$g(N_1 \cap N_2)g^{-1} = \{ghg^{-1} \mid h \in N_1 \cap N_2\}$$

vs.

$$gN_1 g^{-1} \cap gN_2 g^{-1} = \{ghg^{-1} \mid h \in N_1\} \cap \{ghg^{-1} \mid h \in N_2\}.$$

Q3 Suppose that $G = \langle g \rangle$, and let $fH \in G/H$ be given. Since G is cyclic, $\exists\, k \in \mathbb{Z}$ such that $f = g^k$. But then

$$fH = g^k H = (gH)^k$$

so that every element in G/H is a power ~~generator~~ of gH, thus G/H is cyclic.

Q4 a) Define a map $\phi : \mathbb{R}/H \to G$ by

$$\phi(\theta + H) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

b) Suppose that $\alpha + H = \beta + H$. Then $\alpha = \beta + 2\pi k$ for some $k \in \mathbb{Z}$. Therefore

$$\phi(\alpha + H) = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} = \begin{bmatrix} \cos(\beta+2\pi k) & -\sin(\beta+2\pi k) \\ \sin(\beta+2\pi k) & \cos(\beta+2\pi k) \end{bmatrix}$$

$$= \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix}$$

$$= \phi(\beta + H),$$

so ϕ is well-defined.

c) ϕ is clearly surjective. To see it is injective, suppose $\phi(\alpha + H) = \phi(\beta + H)$. Then

$$\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} = \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix}$$

So $\cos(\alpha) = \cos(\beta)$ and $\sin(\alpha) = \sin(\beta)$. To solve this, observe that this gives

$$\cos(\alpha)\cos(\beta)+\sin(\alpha)\sin(\beta)=\cos^2\alpha+\sin^2\alpha=1$$

$$\Rightarrow \cos(\alpha-\beta)=1, \quad \text{so } \alpha-\beta=2k\pi$$

$$\Rightarrow \alpha=\beta+2k\pi.$$

Therefore $\alpha+H=\beta+H$.

d) To see that ϕ respects the group operation, observe that

$$\phi(\alpha+H)\cdot\phi(\beta+H)=\begin{bmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{bmatrix}\begin{bmatrix}\cos\beta & -\sin\beta\\ \sin\beta & \cos\beta\end{bmatrix}$$

$$=\begin{bmatrix}\cos\alpha\cos\beta-\sin\alpha\sin\beta & -\cos\alpha\sin\beta-\sin\alpha\cos\beta\\ \cos\beta\sin\alpha+\cos\alpha\sin\beta & \cos\alpha\cos\beta-\sin\alpha\sin\beta\end{bmatrix}$$

$$=\begin{bmatrix}\cos(\alpha+\beta) & -\sin(\alpha+\beta)\\ \sin(\alpha+\beta) & \cos(\alpha+\beta)\end{bmatrix}.$$

while

$$\phi((\alpha+H)+(\beta+H))=\phi((\alpha+\beta)+H)=\begin{bmatrix}\cos(\alpha+\beta) & -\sin(\alpha+\beta)\\ \sin(\alpha+\beta) & \cos(\alpha+\beta)\end{bmatrix}.$$