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§3.2 Exponentials and Logs.

You are likely familiar with the rules:

If $a>0$, then

(i) $a^0 = 1$

(ii) $a^n = \underbrace{a \cdot a \cdot \ldots \cdot a}_{n \text{ times}}$ $\Bigg\}$ n positive integer

(iii) $a^{-n} = \dfrac{1}{a^n}$

(iv) $a^{m/n} = \sqrt[n]{a^m}$, n pos integer, m any integer

These rules almost define a function $f(x) = a^x$, but not quite, because we would like to know how to compute numbers like $f(\pi) = a^\pi$ as well. (irrationals).

For these numbers, we will use a sequence of approximating rationals to say what the value should be, e.g. if $x = \pi = 3.1415926\ldots$ then

$x_1 = f(3) = a^3$

$x_2 = f(3.1) = a^{31/10}$

$x_3 = f(3.14) = a^{314/100}$

$\vdots$

etc,

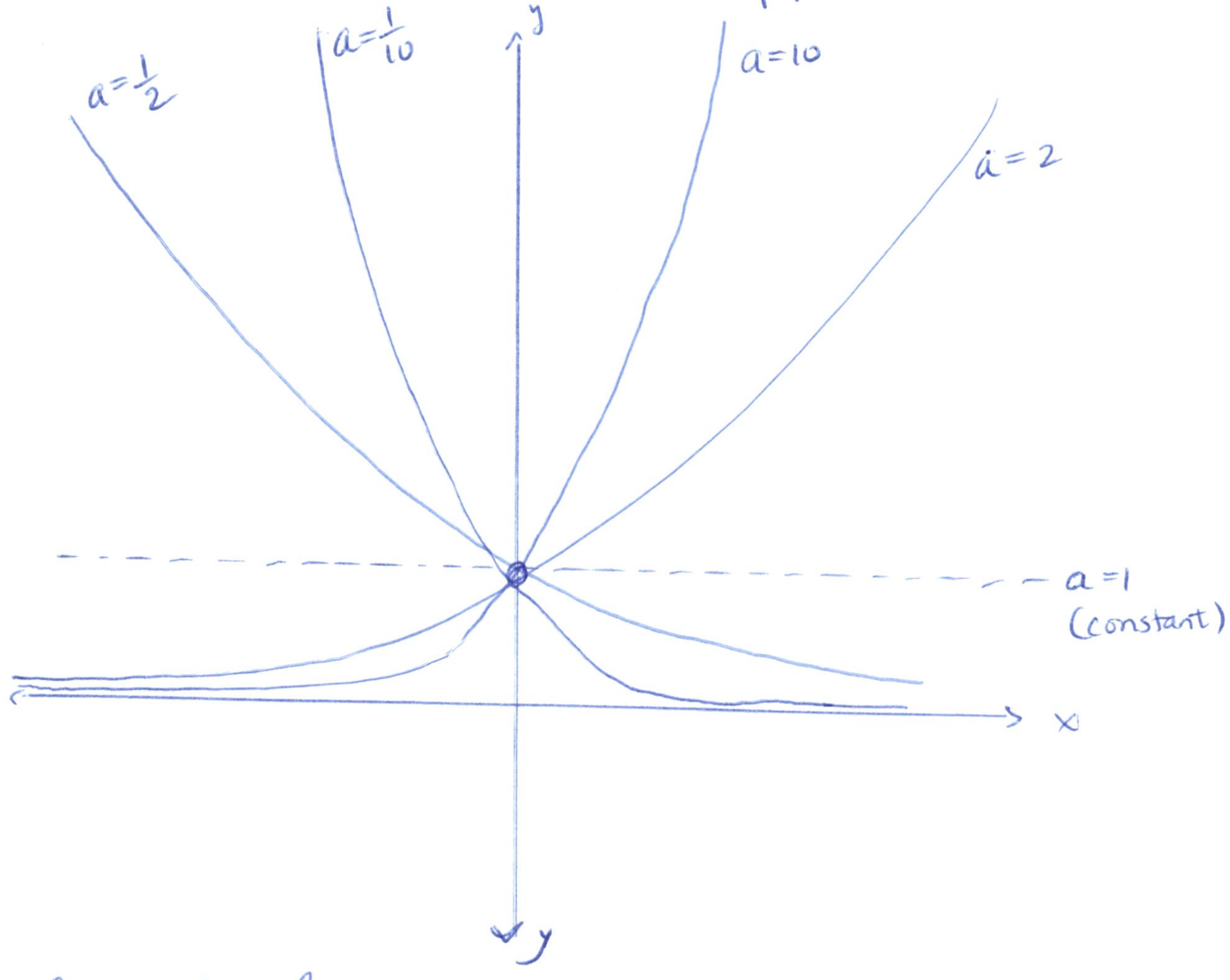
and then we define $f(\pi) = \lim_{n\to\infty} x_n$.

Of course the problem here is that we have not studied sequences and their convergence, so we have to take it as fact (as the book does) that this can work.

The result is a continuous function $f(x) = a^x$ obeying the standard identities (i) – (iv) above, also

(v) $(a^x)^y = a^{xy}$ and (vi) $(ab)^x = a^x b^x$.

The graph, for different a-values, appears below:



Aside from the function $f(x) = (1)^x = 1$, all of the functions above are either increasing/decreasing for different a-values.

Thus there is an inverse function.

<u>Definition</u>: If $a>0$, $a\neq 1$, then $\log_a(x)$ is the inverse of the function a^x, thus

$$\underbrace{\log_a(a^x)=x}_{\text{for all } x} \quad \text{and} \quad \underbrace{a^{\log_a(x)}=x}_{\text{for all } x>0}.$$

Remark: The restriction on the second identity (all $x>0$) is because the range of a^x is $(0,\infty)$, so the domain of $\log_a(x)$ is $(0,\infty)$.

Corresponding to the laws of exponents, we then have the following logarithm rules:

If x, y, a, b are positive and $a, b \neq 1$, then

(i) $\log_a(1)=0$

(ii) $\log_a(xy)=\log_a x+\log_a y$

(iii) $\log_a\left(\frac{x}{y}\right)=\log_a x-\log_a y$

(iv) $\log_a(x^y)=y\log_a x$

(v) $\log_a x=\dfrac{\log_b x}{\log_b a}$

Here is how we would prove them:

Example: Show $\log_a(xy) = \log_a(x) + \log_a(y)$.

Solution: Since the function $f(x) = a^x$ is either increasing or decreasing, it is one-to-one. So

$$f(\log_a(xy)) = a^{\log_a(xy)} = xy$$

and $f(\log_a(x) + \log_a(y)) = a^{\log_a x + \log_a y} = a^{\log_a x} \cdot a^{\log_a y}$

$$= xy$$

forces $\log_a(xy) = \log_a(x) + \log_a(y)$.

Example: Show that $\log_a x = \dfrac{\log_b x}{\log_b a}$.

Solution: Here we must use the rule $\log_a(x^y) = y\log_a(x)$.

Note that $f(x) = a^x$ is a one-to-one function, and so is $g(x) = \log_b x$. Therefore so is their composition $g \circ f(x)$. Now we calculate

$$g \circ f(\log_a x) = g(a^{\log_a x}) = g(x) = \log_b x, \text{ and}$$

$$g \circ f\left(\frac{\log_b x}{\log_b a}\right) = g\left(a^{\frac{\log_b x}{\log_b a}}\right) = \log_b\left(a^{\frac{\log_b(x)}{\log_b a}}\right)$$

used $\log(x^y) = y\log(x)$.

$$= \frac{\log_b(x)}{\cancel{\log_b(a)}} \cdot \cancel{\log_b(a)}$$

$$= \log_b(x).$$

So, since $g \circ f$ is one-to-one, we get

$$\log_a(x) = \frac{\log_b(x)}{\log_b(a)}.$$

<u>Example</u>: Solve $3^{x-1} = 2^x$.

<u>Solution</u>: Choose any base $a > 0$. Then

$$\log_a(3^{x-1}) = \log_a(2^x)$$

$$\Rightarrow (x-1)\log_a(3) = x\log_a(2)$$

$$\Rightarrow (\log_a(3) - \log_a(2))x = \log_a(3)$$

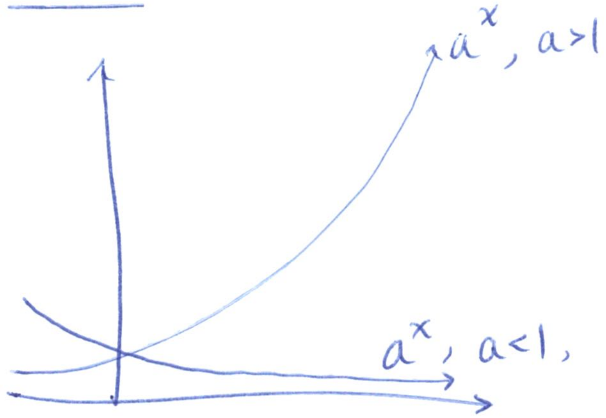
$$\Rightarrow x = \frac{\log_a(3)}{\log_a(3) - \log_a(2)}.$$

Since we have freedom in choosing the base of our logarithms in situations like this, starting next class we will choose a "good" base that we'll be using from here on out.

<u>Exponentials, logs and some infinite limits</u>

As a final note, let's make explicit a few limits that are obvious from the graphs of a^x and $\log_a(x)$.

Recall:



Therefore:

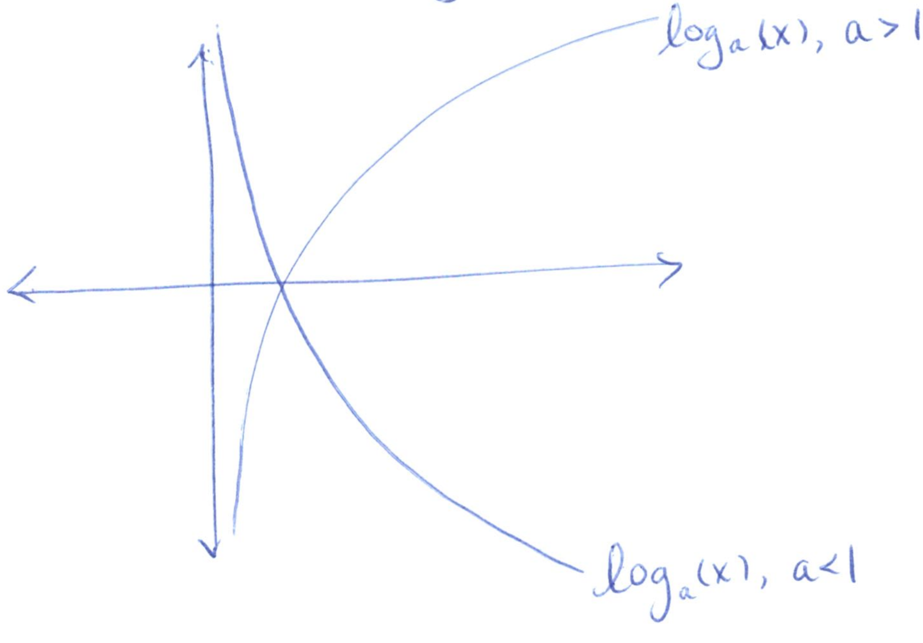
$$a>1 \Rightarrow \lim_{x\to\infty} a^x = \infty,$$

$$\lim_{x\to-\infty} a^x = 0$$

and if $a<1$, then

$$\lim_{x\to\infty} a^x = 0 \quad , \quad \lim_{x\to-\infty} a^x = \infty.$$

Whereas for logarithms



So $$\lim_{x\to\infty} \log_a(x) = \begin{cases} \infty & \text{if } a>1 \\ -\infty & \text{if } 0<a<1 \end{cases}$$

$$\lim_{x\to 0^+} \log_a(x) = \begin{cases} \infty & \text{if } a<1 \\ -\infty & \text{if } a>1. \end{cases}$$

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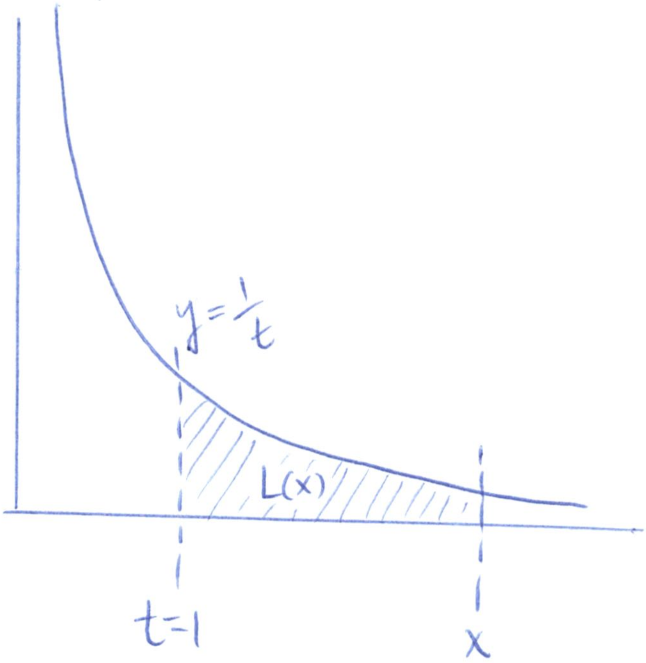
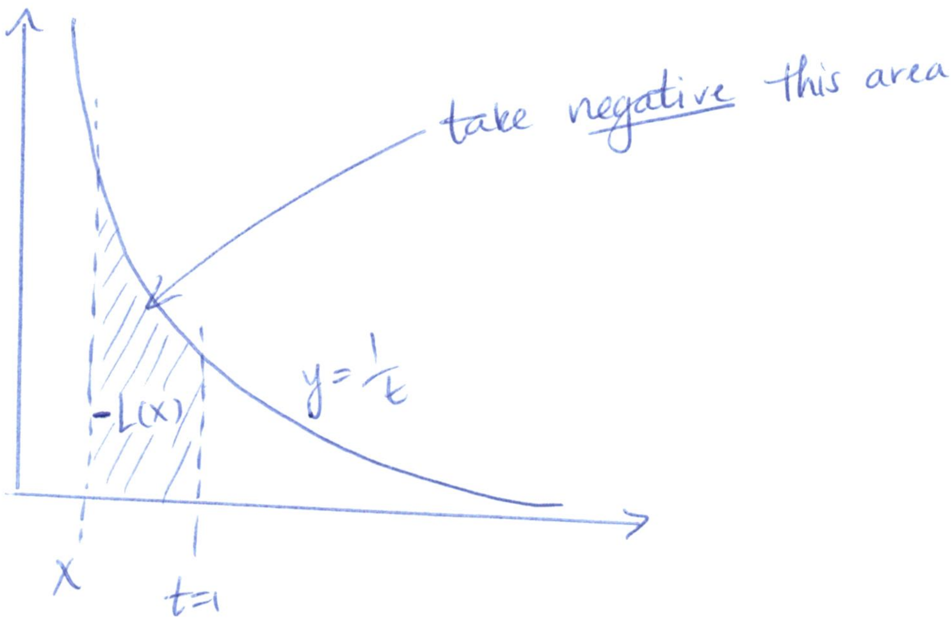
§ 3.3 Natural logs and exponentials.

There is an alternative approach to the material of last class. Instead of defining exponentials first, we can define logarithms first and proceed from there. But how?

What we will do is define an area function $L(x)$ by looking under the curve $y = \frac{1}{t}$, as in the following pictures:



and



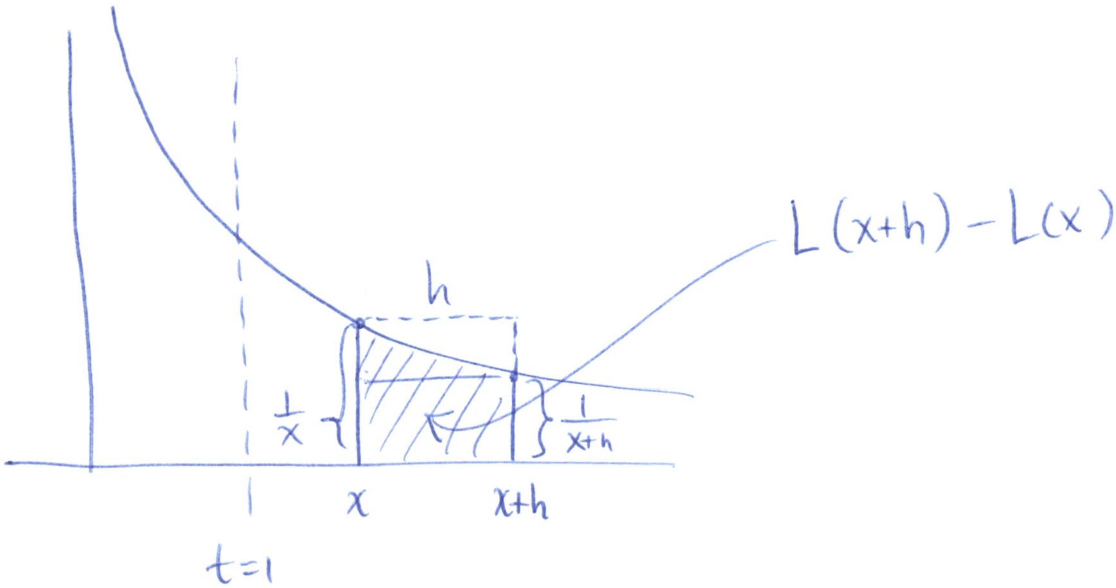
So

$$L(x) = \begin{cases} \text{area under } y=\frac{1}{t} \text{ from } t=1 \text{ to } t=x & x \geq 1 \\ 0 \quad \text{if} \quad x=1. \\ -\text{area under } y=\frac{1}{t} \text{ from } x \text{ to } t=1 \text{ if } x \leq 1 \end{cases}$$

Theorem: If $x > 0$, then $\frac{d}{dx} L(x) = \frac{1}{x}$.

Proof: Consider

$$\lim_{h \to 0} \frac{L(x+h) - L(x)}{h}.$$ From the picture:



we get

$$\frac{h}{x+h} \leq L(x+h) - L(x) \leq \frac{h}{x}$$

small ~~big~~ rectangle area

big rectangle area

So the squeeze theorem gives

$$\lim_{h \to 0} \frac{1}{x+h} \leq \lim_{h \to 0} \frac{L(x+h) - L(x)}{h} \leq \lim_{h \to 0} \frac{1}{x}$$

$$\Rightarrow \quad \frac{1}{x} \leq L'(x) \leq \frac{1}{x}$$

$$\Rightarrow L'(x) = \frac{1}{x}.$$

From this derivative information, we can prove that $L(x)$ must satisfy:

$$\left.\begin{array}{ll}(i) & L(xy) = L(x)+L(y) \\ (ii) & L\left(\frac{x}{y}\right) = L(x) - L(y) \\ (iii) & L(x^r) = r\,L(x).\end{array}\right\}\ \text{i.e, it behaves like a } \underline{\text{logarithm}}.$$

In fact it <u>IS</u> a logarithm, here is how to see this. Let 'e' denote the number that satisfies $L(e)=1$. Then observe that (iii) gives

$$L(e^r) = r\,L(e) = r\cdot 1 = r.$$

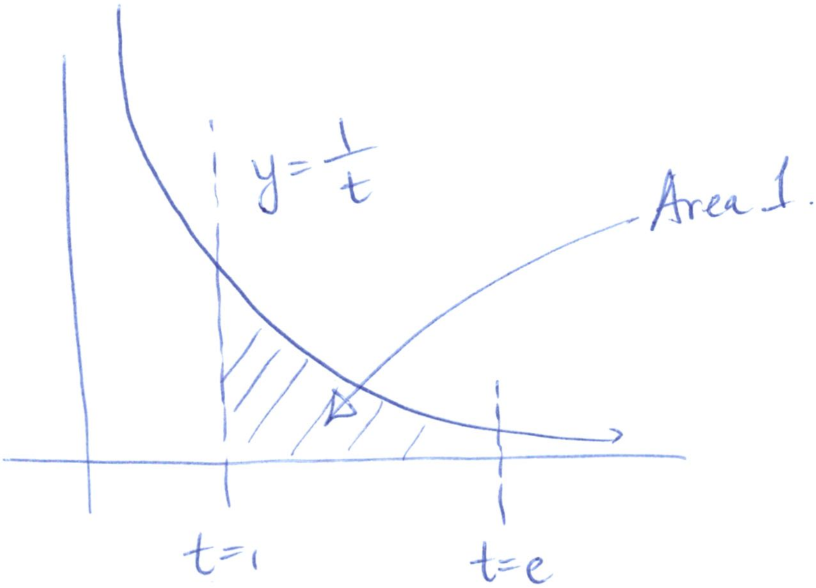
So we can calculate that, by putting $r = \log_e(x)$ for any x:

$$\log_e(x) = L(e^{\log_e x}) = L(x).$$

So the "area under the curve" function is a logarithm, with base e.

From now on we denote this special logarithm by $\ln(x)$, and simply call it "log".

The inverse of $\ln(x)$ will be e^x, and the base e of this function is defined by an area:



It is approximately 2.718.

We already calculated that $\frac{d}{dx}\ln(x) = \frac{1}{x}$, what about $\frac{d}{dx}e^x$? Well, we use implicit differentiation:

$y = e^x \iff x = \ln(y)$ (because ln and e^x are inverses of one another)

$\Rightarrow \frac{d}{dx}(x) = \frac{d}{dx}\ln(y)$

$\Rightarrow 1 = \frac{1}{y}\frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} = y = e^x.$

So we have the rule: $\boxed{\frac{dy}{dx}e^x = e^x}$

From these rulese we can also differentiate all logarithms $\log_a(x)$ and all exponentials a^x.

Since $a^x = e^{x\ln(a)}$ (why? Think $e^{x\ln(a)} = e^{\ln(a^x)} = a^x$, though in reality we are taking this as a definition).

We calculate

$$\frac{d}{dx} a^x = \frac{d}{dx} e^{x\ln(a)}$$

$$= e^{x\ln(a)} \cdot \ln(a) \quad \text{(by the chain rule)}$$

$$= a^x \ln(a).$$

So we have $\boxed{\frac{d}{dx} a^x = a^x \ln(a)}$.

Example: Calculate the derivative of $y = x^\pi - \pi^x$.

Solution: We find

$$y' = \underbrace{\pi x^{\pi-1}}_{\text{power rule}} - \underbrace{\pi^x \ln(\pi)}_{\text{exponential rule.}}$$

note the different rules!

We can also differentiate $\log_a(x)$. Recall that if $f(x)$ and $f^{-1}(x)$ are inverse functions, then

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}.$$ Here, if $f(x) = a^x$ and

$f^{-1}(x) = \log_a(x)$, then

$$\boxed{\frac{d}{dx}\log_a(x) = \frac{1}{a^{\log_a(x)}\cdot \ln(a)} = \frac{1}{x\ln(a)}}$$

From here on we will tend to avoid using $\log_a(x)$ when $a \neq e$, because we can always change base:

$$\log_a(x) = \frac{\ln x}{\ln(a)}$$ (recall this formula from prev. class).

<u>Example</u>: Calculate the derivative of

$$y = \ln(5^x + \ln(x))$$

<u>Solution</u>: By the chain rule,

$$y' = \frac{1}{5^x + \ln(x)} \cdot \left(5^x \ln(5) + \frac{1}{x}\right).$$

<u>Example</u>: Calculate the derivative of $\ln(\ln(\ln(x)))$.

<u>Solution</u>: Again by the chain rule:

$$y' = \frac{1}{\ln(\ln(x))} \cdot \frac{1}{\ln(x)} \cdot \frac{1}{x}.$$

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§3.5 Inverse trig functions.

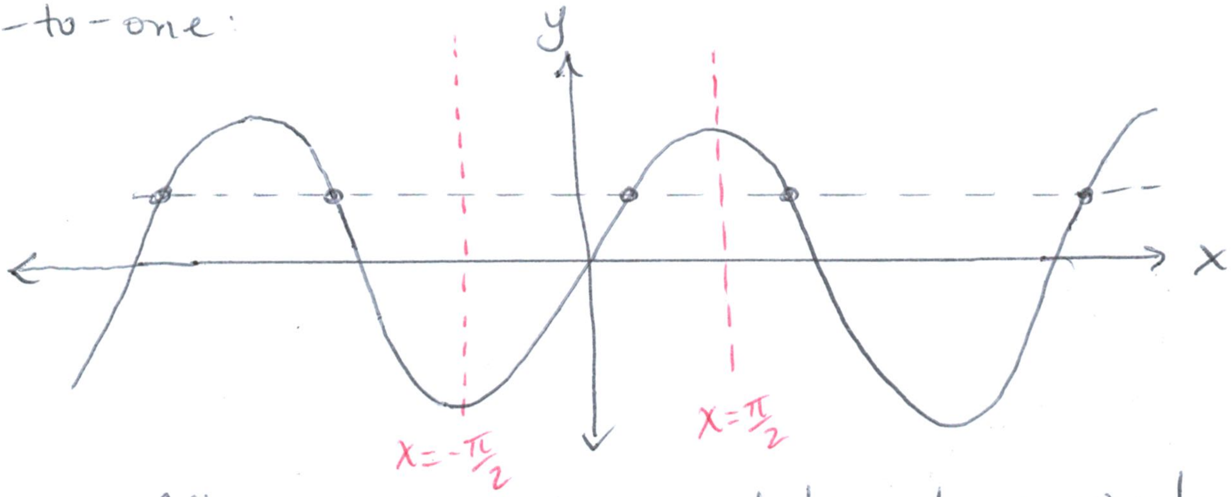
Recall that a function $f(x)$ has an inverse if and only if $f(x)$ is one-to-one. In some cases, we saw that this means an inverse exists only on a restricted domain

For example, $f(x) = \sqrt{6x+5}$ is defined only for $x \geq 0$. It inverse is $f^{-1}(x) = \frac{x^2-5}{6}$, and looks like it might be defined everywhere, but it's not! The defining identities

$$f(f^{-1}(x)) = x \quad \text{and} \quad f^{-1}(f(x))$$

only hold if we restrict to $x \geq 0$.

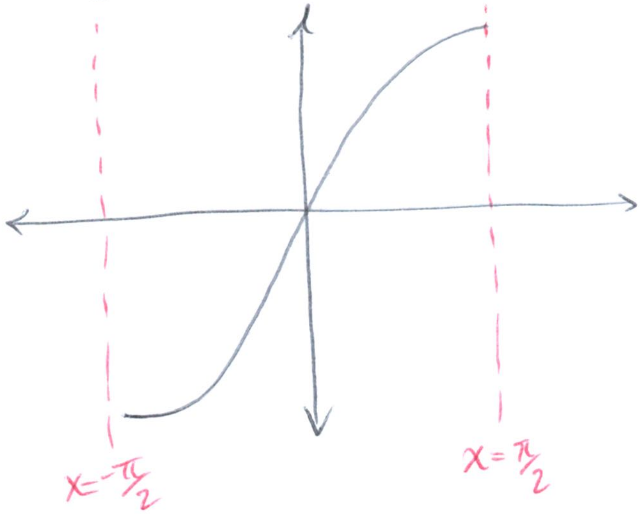
Inverse of $\sin(x)$: The function $\sin(x)$ is definitely not one-to-one:



All marked points are solutions to $\sin(x) = \frac{1}{2}$, for example.

To make the function one-to-one you must restrict the domain.

Our convention will be to restrict to $[\frac{-\pi}{2}, \frac{\pi}{2}]$, where $\sin(x)$ looks like:



On this interval, $\sin(x)$ has an inverse, denoted $\sin^{-1}(x)$.

I.e. $\sin(x) = y$ if and only if $\sin^{-1}(y) = x$
x in $[-\pi/2, \pi/2]$ $\qquad$ y in $[-1, 1]$.

The restriction of only plugging in numbers $[-1, 1]$ to $\sin^{-1}(y)$ comes from the y-values (range) of $\sin(x)$.

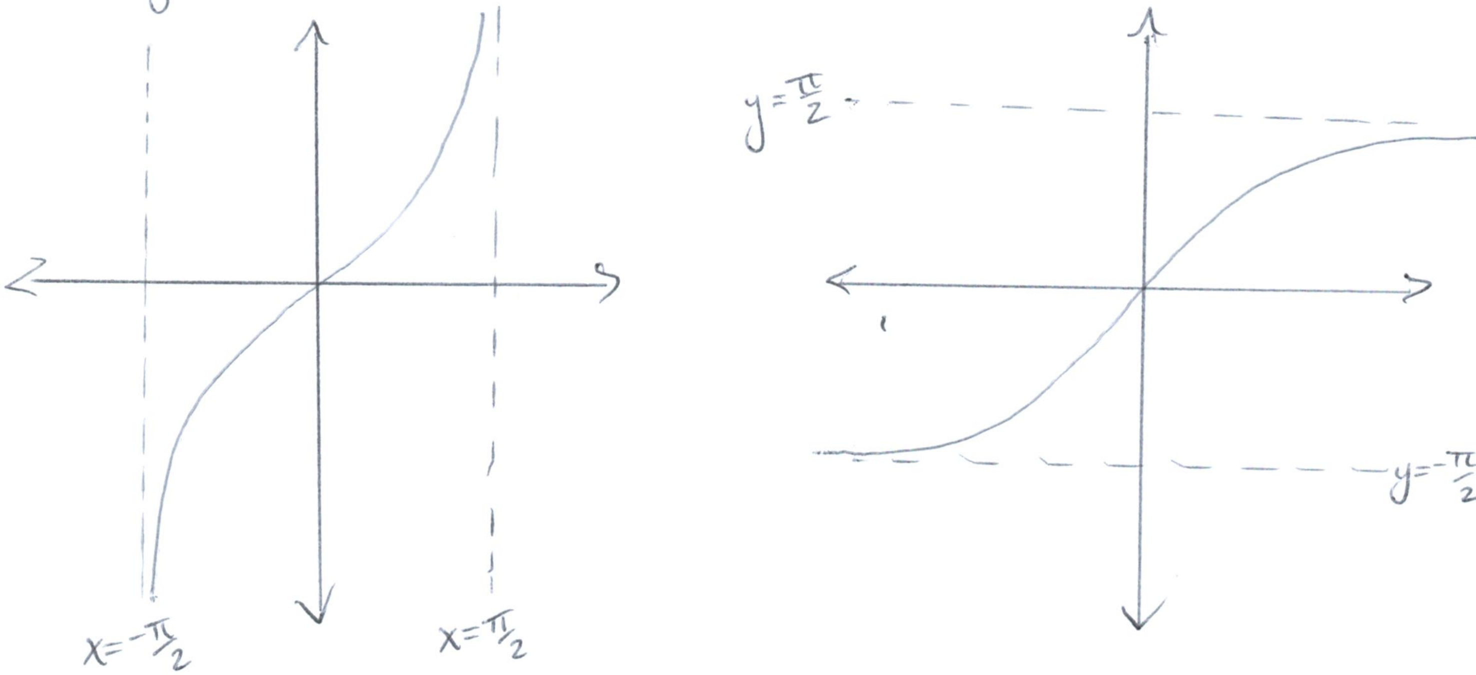
We can similarly define:

$\cos^{-1}(y) = x$ for $\iff$ $\cos(x) = y$
y in $[0, 1]$ $\qquad$ for x in $[0, \pi]$

and

$\tan^{-1}(y) = x$ $\iff$ $\tan(x) = y$ for
(no restriction on y) $\qquad$ x in $[-\pi/2, \pi/2]$.

The graphs of $\tan(x)$ and $\tan^{-1}(x)$ are:



Evaluating inverse trig functions:

Example: Find $\sin^{-1}(\frac{1}{2})$.

Solution: Since $\sin(\frac{\pi}{6}) = \frac{1}{2}$, and $\sin^{-1}(y) = x$ iff $\sin(x) = y$, we know that $\sin^{-1}(\frac{1}{2}) = \pi/6$.

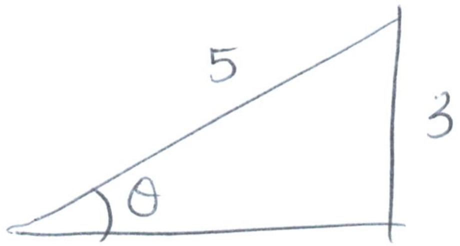
Example: Find $\sin^{-1}(-1)$ and $\sin^{-1}(2)$:

Solution: Since $\sin(\frac{-\pi}{2}) = -1$, $\sin^{-1}(-1) = -\frac{\pi}{2}$. Note that in fact $\sin(\frac{-\pi}{2} + 2k\pi) = -1$ for all integers k, but $k=0$ gives the only value in our restricted domain $[\frac{-\pi}{2}, \frac{\pi}{2}]$ for $\sin(x)$.

On the other hand, 2 is not in the domain of $\sin^{-1}(x)$, so $\sin^{-1}(2)$ is not defined.

Example: Simplify $\cos(\sin^{-1}(3/5))$.

Solution: We build a triangle. $\sin^{-1}(3/5)$ is an angle θ that fits into the triangle



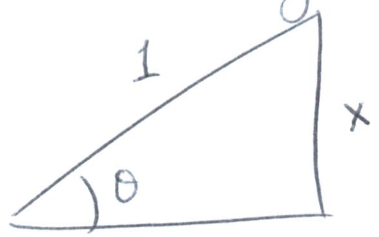
, because we know $\sin\theta = \frac{opp}{hyp} = 3/5$.

By the Pythagorean theorem the length of the unlabeled side is $\sqrt{5^2-3^2} = \sqrt{16} = 4$, so then $\cos(\theta) = \frac{adj}{hyp} = \frac{4}{5}$.

So $\cos(\sin^{-1}(3/5)) = \frac{4}{5}$.

Example: Simplify $\tan(\sin^{-1}(x))$.

Solution: $\sin^{-1}(x)$ is an angle θ that fits into the triangle



$\sin(\theta) = \frac{opp}{hyp} = \frac{x}{1} = x$.

The length of the third side is $\sqrt{1-x^2}$, so

$$\tan(\sin^{-1}(x)) = \frac{opp}{adj} = \frac{x}{\sqrt{1-x^2}}.$$

Derivatives of inverse trig functions:

To find the derivative of an inverse trig function, we use implicit differentiation.

Example: If $y = \sin^{-1}(x)$, then $x = \sin(y)$.

So $$\frac{d}{dx}(x) = \frac{d}{dx}(\sin(y))$$

$$\Rightarrow \quad 1 = \cos(y)\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\cos(y)}.$$

But $\cos(y) = \cos(\sin^{-1}(x))$ simplifies to $\frac{1}{\sqrt{1-x^2}}$, so

$$\boxed{\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}}.$$

If $y = \tan^{-1}(x)$ then $x = \tan(y)$, so

$$\frac{d}{dx}(x) = \frac{d}{dx}\tan(y)$$

$$\Rightarrow 1 = \sec^2(y)\frac{dy}{dx} \Rightarrow 1 = (1 + \underbrace{\tan^2(y)}_{=x^2})\frac{dy}{dx}$$

$$\cancel{\Rightarrow \cos^2(y) = \frac{dy}{dx}} \qquad \Rightarrow \boxed{\frac{dy}{dx} = \frac{1}{1+x^2}}$$

And, via similar methods, we get

$$\frac{d}{dx}\cos^{-1}(x) = -\frac{1}{\sqrt{1-x^2}}.$$

We can also define $\sec^{-1}(x)$, $\csc^{-1}(x)$, and $\cot^{-1}(x)$ by appropriately restricting the domains of these functions and then calculate their derivatives. See pages 197-198.

Example: Evaluate $\int \frac{1}{x^2+a^2}\,dx$.

Solution: To do this, we need to find a function whose derivative is $\frac{1}{x^2+a^2}$. Let's try $\tan^{-1}\left(\frac{x}{a}\right)$.

We get: $$\frac{d}{dx}\tan^{-1}\left(\frac{x}{a}\right) = \frac{1}{1+\left(\frac{x}{a}\right)^2}\cdot\frac{1}{a} \quad \text{(chain rule)}.$$

$$= \frac{1}{a\left(1+\frac{x^2}{a^2}\right)}$$

But $\frac{1}{a\left(1+\frac{x^2}{a^2}\right)} = \frac{a}{a^2+x^2}$ (Check this!).

Therefore $\frac{d}{dx}\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) = \frac{1}{a^2+x^2}$.

So $$\boxed{\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C.}$$