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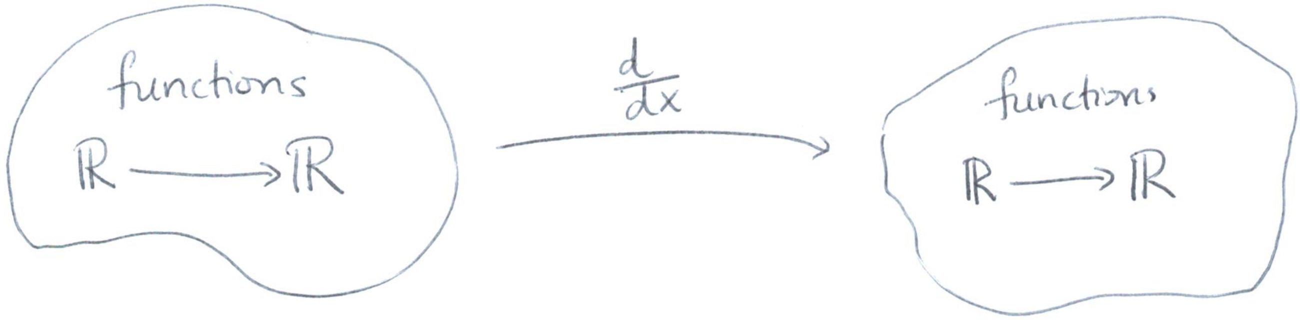
§ 2.10 Antiderivatives + DE's.

Taking antiderivatives is the process of going backwards from a function $f'(x)$ to the original function $f(x)$.

Definition: An antiderivative of a function $f(x)$ on an interval I is a function $F(x)$ satisfying

$$F'(x) = f(x) \text{ for } x \text{ in } I.$$

One way of thinking of antiderivatives is this:



Namely, $\frac{d}{dx}$ (taking the derivative) is like a function which takes in a function $f(x)$ and produces a new one, $\frac{d}{dx} f(x)$. Finding an antiderivative means to find a function $F(x)$ which is sent to $f(x)$ by $\frac{d}{dx}$.

Remark: $\frac{d}{dx}$ is not onto! I.e there are plenty of functions that do not have antiderivatives.

<u>Notation</u>: If $F(x)$ is an antiderivative of $f(x)$, we write

$$\int f(x)\,dx = F(x) + C$$

constant

the "$+C$" is meant to indicate that every other function $G(x) = F(x) + C$ is also an antiderivative of $f(x)$, because

$$\frac{d}{dx}(G(x)) = \frac{d}{dx}\left(F(x)+c\right) = \frac{d}{dx}F(x) + 0 = f(x).$$

In fact, we have:

<u>Theorem</u>: If $F(x)$ is an antiderivative of $f(x)$, then any other antiderivative $G(x)$ of $f(x)$ is equal to $F(x)+C$ for some constant C.

<u>Proof</u>: Requires the Mean Value Theorem and is beyond the scope of this class.

This ends our "theoretical" discussion on antiderivatives. What of the pactical side? I.e., how to calculate $F(x)$ given $f(x)$? For each ~~de~~ differentiation rule, there is a corresponding "antidifferertiation" rule:

(i) $\int C_0\,dx = C_0 x + C$ (C_0 any constant)

(ii) $\displaystyle\int x^n \, dx = \frac{x^{n+1}}{n+1} \quad (n \neq -1)$

(iii) $\displaystyle\int \sin x \, dx = -\cos x + C$

(iv) $\displaystyle\int \cos x \, dx = \sin x + C.$

There are a few other rules, see page 150 for a list.

Remarks ① Note that since

$$\frac{d}{dx}(f(x) \pm g(x)) = \frac{df}{dx} \pm \frac{dg}{dx}$$

the same is true for antiderivatives:

$$\int f(x) \pm g(x) \, dx = \int f(x) \, dx \pm \int g(x) \, dx$$

also

$$\frac{d}{dx} c f(x) = c \frac{df}{dx} \text{ implies } \int c f(x) \, dx = c \int f(x) \, dx.$$

② There is not really an equivalent of the product, quotient or chain rule. This makes taking antiderivatives harder than taking derivatives.

Example:

Find the antiderivatives of $\dfrac{2x^{3/2} + \sqrt{x}}{x^2}$.

Solution: We compute:

$$\int \frac{2x^{3/2} + \sqrt{x}}{x^2}\,dx = \int \frac{2}{\sqrt{x}} + x^{-3/2}\,dx$$

$$= 2\int x^{-1/2}\,dx + \int x^{-3/2}\,dx$$

$$= 2\,\frac{x^{1/2}}{1/2} + \frac{x^{-1/2}}{-1/2} + C$$

note: since C is an arbitrary constant we only put one "C" here (ie. $C_1 + C_2$ can be replaced with a single C).

$$= 4\sqrt{x} - \frac{1}{2\sqrt{x}} + c$$

Applications of antiderivatives

Antiderivatives will turn out to be useful for many, many things (computing areas, for one). We will touch on a basic use: Solving differential equations.

A differential equation (DE) is an equation involving derivatives, e.g.

$$x^2 \cdot \frac{dy}{dx} = \cos(x) + x^3$$

A solution to a DE is a formula for $y(x)$ that makes the given equation true. E.g if we are asked to solve

$$\frac{dy}{dx} = 3x^2 - 1$$

then $y(x) = x^3 - x + C$ is a solution for any value of C. If we leave C arbitrary, it's called the "general solution" to the given DE.

Example: Solve

$$y' = \frac{3 + 2x^2}{x^2}.$$

Solution: If $y' = \frac{3+2x^2}{x^2}$, then

$$\int y'\,dx = \int \frac{3+2x^2}{x^2}\,dx = \int \frac{3}{x^2}\,dx + \int 2\,dx$$

$$= -\frac{3}{x} + 2x + C$$

is a solution.

A particular solution to a DE is when we are told a value of the derivatives that allows us to solve for C, as in the following example.

Example: Solve $y' = x - 2$, $y(0) = 3$.

Solution: If $y' = x - 2$ then

$$\int y' \, dx = \int x - 2 \, dx = \frac{x^2}{2} - 2x + C$$ is a general

solution to the DE. But $y(0) = 3$ means we must

have $$3 = y(0) = \frac{0^2}{2} + 2(0) + C$$

$$\Rightarrow C = 3.$$

So then $y = \frac{x^2}{2} - 2x + 3$. is a particular

solution to the given DE.

Example: Solve $y' = \frac{3 + 2x^2}{x^2}$, $y(-2) = 1$.

Solution: We already found a general solution of

$y(x) = \frac{-3}{x} + 2x + C$. Using $y(-2) = 1$ we get

$$1 = \frac{-3}{-2} + 2(-2) + C$$

$$\Rightarrow 1 = \frac{3}{2} + 4 + C \Rightarrow C = \frac{7}{2}.$$

So $y(x) = \frac{-3}{x} + 2x + \frac{7}{2}$ is a particular solution.

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§2.11 Velocity and Acceleration, applications.

We already saw that if $x(t)$ denotes the position of an object at time t (along the x-axis, say) then

$$v(t) = \frac{dx}{dt} = x'(t) \quad \text{is velocity}$$

$$a(t) = \frac{dv}{dt} = v'(t) = x''(t) \quad \text{is acceleration.}$$

It turns out that with our knowledge of simple DEs and derivatives we can analyze many physical situations involving these quantities.

Example: An object is thrown upwards from the roof of a building 10m tall. It rises then falls back to the ground, with its height at time t given by $y(t) = -4.9t^2 + 8t + 10$ (meters).

(i) What is its maximum height?

(ii) What is its speed when it strikes the ground?

Solution: (i) We compute the velocity and find

$$v(t) = -9.8t + 8 \quad (\text{meters/sec}).$$

The object rises when $v(t) > 0$, falls when $v(t) < 0$, and so is at a max when

$$-9.8t + 8 = 0$$

$$\Rightarrow t = \frac{8}{9.8}.$$

(Alternatively: We look for a line tangent to the curve $y(t)$ having zero slope, as we saw that these correspond to maxes/mins in many cases).

Thus the max height is

$$y\left(\frac{8}{9.8}\right) = -4.9\left(\frac{8}{9.8}\right)^2 + 8\left(\frac{8}{9.8}\right) + 10 \simeq 13.3\,m.$$

(ii) We know the object strikes the ground when $y(t) = 0$, so we solve

$$-4.9t^2 + 8t + 10 = 0$$

$$\Rightarrow t = \frac{-8 \pm \sqrt{64+196}}{-9.8}$$

producing two solutions, one positive and one negative. We only want the positive, ie. $t \doteqdot 2.462$.

At this time,

$$v(2.462) = -9.8(2.462) + 8 = -16.1\,m/s.$$

In this last example we know the position function and deduced properties of the ~~the~~ derivative.

We can also go the other way:

<u>Example</u>: A free falling object, with air resistance neglected, begins from a height y_0 with an initial velocity of v_0. What is its equation of motion (i.e., its position function?)

<u>Solution</u>: Acceleration due to gravity is approximately $-9.8\,m/s^2$ (negative because it's a <u>downward</u> acceleration). So if $y(t)$ is the position function, this says:

$$y''(t) = -9.8.$$

If we take the "beginning" of the object's fall to be time $t=0$, we also have:

$$y'(0) = v_0, \quad y(0) = y_0.$$

Thus we have:

$$y'(t) = \int y''(t)\,dt = \int -9.8\,dt = -9.8t + C_1.$$

Here, the constant C_1 is determined by the condition

$$y'(0) = v_0 \Rightarrow 9.8(0) + C_1 = v_0$$
$$\Rightarrow C_1 = v_0.$$

So $y'(t) = -9.8t + v_0.$

Then $y(t) = \int y'(t)\,dt = \int -9.8t + v_0\,dt$

$$= -4.9t^2 + v_0 t + C_2$$

And the condition $y(0) = y_0$ gives us

$$y_0 = y(0) = -4.9(0) + V_0(0) + C_2$$
$$\Rightarrow C_2 = y_0.$$

Therefore $y(t) = -4.9t^2 + v_0 t + y_0.$

This sort of scenario also applies whenever acceleration is constant.

<u>Example</u>: Suppose a car is travelling at X_0 km/h and is capable of decelerating at $2.5\,m/s^2$. How far, in terms of the initial speed X_0, does the car need to stop?

<u>Solution</u>: Let $t=0$ be the time when the brakes are first applied, and let $s(t)$ denote the position of the car at time t. We know

$$s''(t) = -2.5\,m/s^2$$

and therefore velocity is

$$s'(t) = \int -2.5\,dt = -2.5t + C_1$$

where C_1 is determined by the fact that the car starts at x_0 km/h $\approxeq 0.28 x_0$ m/s.

$$\Rightarrow 0.28x_0 = s'(0) = -2.5(0) + C_1$$

$$\Rightarrow C_1 = 0.28x_0$$

Therefore
$$s'(t) = -2.5t + 0.28x_0 .$$

From this, we can see that the car will stop after:

$$0 = -2.5t + 0.28x_0$$

$$\Rightarrow t = \frac{0.28x_0}{2.5} \text{ seconds.}$$

The distance it will have travelled is found by computing $s(t)$ and plugging in this value.

$$s(t) = \int -2.5t + 0.28x_0 \, dt$$

$$= -1.25t^2 + (0.28x_0)t + C_2 .$$

If we choose the spot where the car initially applies its brakes to be distance 0, we have $s(0) = 0$ and so

$$s(t) = -1.25t^2 + (0.28x_0)t.$$

Thus the car stops a distance of

$$s\left(\frac{0.28x_0}{2.5}\right) = -1.25\left(\frac{0.28x_0}{2.5}\right)^2 + \left(\frac{0.28x_0}{2.5}\right)\cdot(0.28x_0)$$

$= 0.01568\, x_0^2$.

So, for example, a car going 50 km/hr will stop in 39.2 m (a bit far, I admit) A car going 100 km/hr will stop in 156.8 m.

<u>Note</u>: Stopping distance depends on the square of the initial velocity x_0.

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§3.1 Inverse functions

This lecture is largely review and prep for logs/exponentials.

Recall that a function is one-to-one if $f(x_1) = f(x_2)$ implies $x_1 = x_2$, meaning $f(x) = y$ has a unique solution for x.

Example: The function $f(x) = \sqrt{x}$ is one-to-one.

If $f(x_1) = f(x_2)$ then $\sqrt{x_1} = \sqrt{x_2}$

$$\Rightarrow x_1 = x_2 \text{ (square both sides)}$$

On the other hand, $f(x) = x^2$ is not one-to-one since $(-1)^2 = 1^2 = 1$.

Definition: If $f(x)$ is a function, its inverse function is a function $g(x)$ satisfying

$$g(f(x)) = x \quad \text{and} \quad f(g(x)) = x.$$

Instead of $g(x)$, we usually denote it by $f^{-1}(x)$.

Remark: If $f(x)$ is one-to-one, then it has an inverse. Whenever $f(x) = y$, we define the inverse by the formula

$$f^{-1}(y) = x$$

This formula "makes sense", because if f is one-to-one then there's a unique x with $f(x) = y$.

Recall the following method for finding inverse functions:

If $f(x)$ is given, you:

1. Replace $f(x)$ with y
2. Replace every x with a y and y with an x.
3. Solve for y, relpace y with $f^{-1}(x)$.

Example: Find the inverse of $f(x) = \sqrt{6x+5}$.

Solution 1. Write $y = \sqrt{6x+5}$

2. Change to $x = \sqrt{6y+5}$

3. Solve:

$$x^2 = 6y+5$$

$$\Rightarrow y = \frac{x^2-5}{6}.$$

Therefore $f^{-1}(x) = \frac{x^2-5}{6}$.

There is one final step in this case: Since the range of $f(x)$ is $x \geq 0$, we should point out that the domain of $f^{-1}(x)$ is only $[0, \infty)$, even though every real number works in the formula $\frac{x^2-5}{6}$.

This domain restriction is necessary! Suppose we allow an x outside of $[0, \infty)$, say $x = -1$.

Then $f(f^{-1}(-1)) = f\left(\frac{(-1)^2-5}{6}\right)$

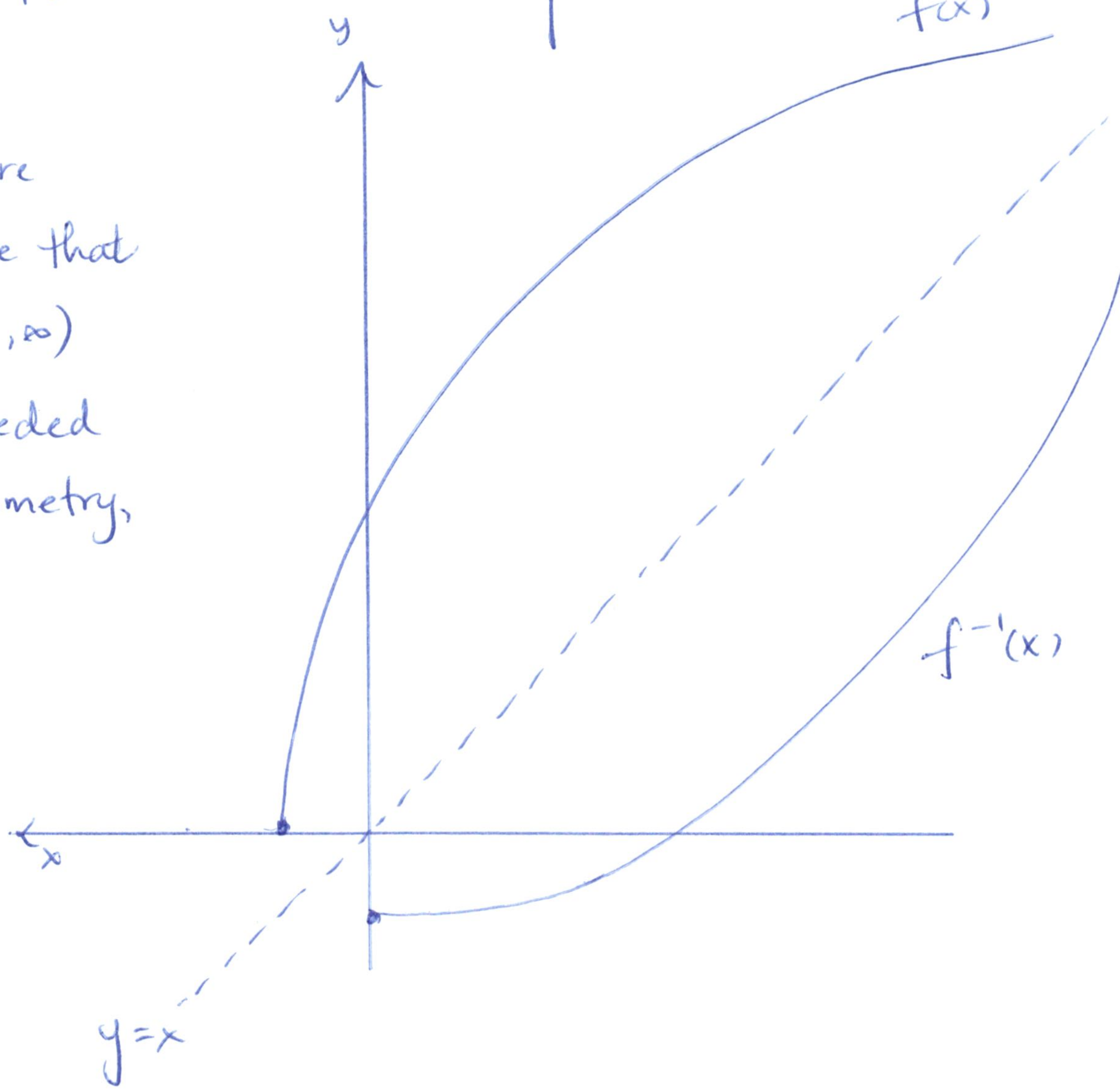
$= f\left(-\frac{2}{3}\right)$

$= \sqrt{6\left(-\frac{2}{3}\right)+5} = \sqrt{5-4} = 1$

So the defining identity of $f(f^{-1}(x)) = x$ fails for points outside of $[0,\infty)$. Hence the restricted domain.

Graphically, the function $f(x)$ is related to its inverse $f^{-1}(x)$ by reflection in the line $y=x$:

From the picture we can also see that the domain $[0,\infty)$ for $f^{-1}(x)$ is needed to preserve symmetry, since $f(x) \geq 0$



Sometimes we also need to restrict the domain of our original function $f(x)$ (not just its inverse), to ensure that $f(x)$ is one-to-one.

Example: Consider $f(x) = x^2$. It is not one-to-one so it cannot have an inverse. Yet we know that $\sqrt{x}$ and x^2 should "cancel" in some sense. Precisely:

If $x \geq 0$ then

$$\sqrt{x^2} = |x| = x$$

and $(\sqrt{x})^2 = x$

So on $[0, \infty)$ the function $f(x) = x^2$ has $f^{-1}(x) = \sqrt{x}$ as an inverse.

The derivative of $f^{-1}(x)$ can be calculated from the defining identities using the chain rule.

$$f(f^{-1}(x)) = x$$

$$\Rightarrow \frac{d}{dx}\left(f(f^{-1}(x))\right) = \frac{d}{dx} x$$

$$\Rightarrow f'(f^{-1}(x)) \cdot \frac{d}{dx} f^{-1}(x) = 1$$

$$\boxed{\Rightarrow \frac{d}{dx} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}}$$

as long as $f'(x)$ is never zero.

For example, if $f(x)$ is an increasing ($f'(x) > 0$) or decreasing ($f'(x) < 0$) function then this formula holds.

<u>Example</u>: Consider $f(x) = 3x^5 + x^3 + x$. Does $f^{-1}(x)$ exist? If yes, what is $(f^{-1})'(5)$?

<u>Solution</u>: Note that since $f'(x)$ is an even function:

$$\left(\begin{aligned} f'(x) &= 15x^4 + 3x^2 + 1 \\ &\quad\;\; \parallel \qquad\;\; \parallel \\ \text{and } f'(-x) &= 15(-x)^4 + 3(-x)^2 + 1 \text{ are equal} \end{aligned} \right)$$

we know that $f'(x) > 0$ for all x in $\mathbb{R}$, since $f'(x) = 15x^4 + 3x^2 + 1$ is certainly positive for all positive x.

Therefore $f(x)$ is increasing, so it must be one-to-one (why?), and ~~that~~ therefore $f^{-1}(x)$ exists.

We can't find a formula for $f^{-1}(x)$, but since $f(1) = 3(1)^5 + 1^3 + 1 = 5$, we know $f^{-1}(5) = 1$. Therefore

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}, \text{ with } x = 5 \text{ gives}$$

$$(f^{-1})'(5) = \frac{1}{15(1)^4 + 3(1)^2 + 1} = \frac{1}{19}.$$

(Difficult).

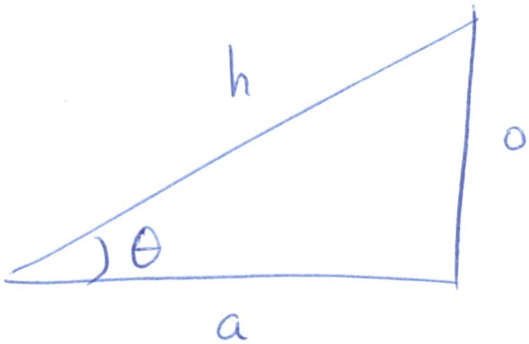
Example: We will eventually introduce and study inverse trig functions, but we can already use this rule to say what their derivatives must be!

For example, if $\sin^{-1}(x)$ is a function with

$\sin(\sin^{-1}(x)) = x$ and $\sin^{-1}(\sin(x))$ on some domain, then

$$\frac{d}{dx}\left(\sin^{-1}(x)\right) = \frac{1}{\cos\left(\sin^{-1}(x)\right)} \qquad \left(\text{since } \frac{d}{dx}\sin x = \cos x\right).$$

From the triangle, observe that:



$$\sin\theta = \frac{o}{h} = x$$

$$\Rightarrow \sin^{-1}\left(\frac{o}{h}\right) = \theta$$

$$\Rightarrow \cos\left(\sin^{-1}\left(\frac{o}{h}\right)\right) = \cos(\theta)$$

$$= \frac{a}{h}.$$

$$\Rightarrow \frac{d}{dx}\sin^{-1}(x) = \frac{1}{\frac{a}{h}}$$

But from the fact that $h^2 = o^2 + a^2$, we can check that

$$\frac{a}{h} = \sqrt{1 - \frac{o^2}{h^2}} = \sqrt{1 - x^2},$$ so

$$\frac{d}{dx}\left(\sin^{-1}(x)\right) = \frac{1}{\sqrt{1-x^2}}.$$