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§2.3 Derivative rules.

If $f(x)$ and $g(x)$ are differentiable at x, and c is a constant, then:

(i) $(f(x) \pm g(x))' = f'(x) \pm g'(x)$ or $\frac{d}{dx}(f+g) = \frac{df}{dx} + \frac{dg}{dx}$

(ii) $(cf(x))' = cf'(x)$ or $\frac{d}{dx}(cf) = c\frac{df}{dx}$.

Hereis an example of how one proves such rules.

Example: Show that $(f(x)+g(x))' = f'(x)+g'(x)$.

Solution: By definction,

$$(f(x)+g(x))' = \lim_{h\to 0} \frac{(f+g)(x+h) - (f+g)(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)+g(x+h) - (f(x)+g(x))}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)-f(x) + g(x+h)-g(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} + \lim_{h\to 0} \frac{g(x+h)-g(x)}{h}$$

$$= f'(x)+g'(x).$$

Thus the "sum rule" holds.

With the rules (and the power rule from last day) we can now differentiate any polynomial.

<u>Example</u>: If $f(x) = 6x^3 - 2x^2 - 1$, compute $f'(x)$.

<u>Solution</u>:

$$f'(x) = 6\cdot(3x^2) - 2(2x) - 0$$
$$= 18x^2 - 4x.$$

<u>Example (Theorem)</u>: Calculate a formula for the derivative of a product of functions.

<u>Solution</u>: Suppose functions $f(x)$ and $g(x)$ each have a derivative at every point in some interval I. Then

$$(fg)'(x) = \lim_{h\to 0} \frac{(fg)(x+h) - (fg)(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}.$$

$$= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h}.$$

$$= \lim_{h\to 0} g(x+h)\left(\frac{f(x+h) - f(x)}{h}\right) + \lim_{h\to 0} f(x)\left(\frac{g(x+h) - g(x)}{h}\right)$$

Now we are sort of stuck. What we would like to do is this:

$$= \underbrace{\lim_{h\to 0} g(x+h)}_{= g(x)} \underbrace{\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}}_{= f'(x)} + f(x) \underbrace{\lim_{h\to 0} \frac{g(x+h)-g(x)}{h}}_{= g'(x)}$$

$$= g(x) f'(x) + f(x) g'(x).$$

Then we arrive at the product rule:

$$\boxed{(fg)'(x) = g(x)f'(x) + f(x)g'(x).}$$

However, there was a small (but important!) gap in the last argument. The gap was this step:

$$\lim_{h\to 0} g(x+h) = g(x)$$

Here we evaluated the limit by plugging in $h=0$. This is only possible if $g(x)$ is a continuous function, and we only know that $g(x)$ is differentiable. So the missing step in our proof is:

Theorem: If $g(x)$ is differentiable at x_0, then it is continuous at x_0.

Proof: If $g(x)$ is differentiable at x_0, then

$$\lim_{h\to 0} \frac{g(x_0+h) - g(x_0)}{h} \text{ exists.}$$

The limit rules give us:

$$\lim_{h\to 0}\left(g(x_0+h)-g(x_0)\right) = \lim_{h\to 0}\left(\frac{g(x_0+h)-g(x_0)}{h}\right)\cdot h$$

$$= \lim_{h\to 0}\frac{g(x_0+h)-g(x_0)}{h}\cdot\lim_{h\to 0} h$$

$$= g'(x_0)\cdot 0 = 0.$$

So $\lim_{h\to 0}\left(g(x_0+h)-g(x_0)\right)=0$, or

$$\lim_{h\to 0} g(x_0+h) = \lim_{h\to 0} g(x_0) = g(x_0),$$

which is exactly saying that g is continuous at x_0.
So we've filled the missing step in our proof!

We also have a quotient rule:

$$\left(\frac{f}{g}\right)'(x) = \frac{g(x)f'(x)-f(x)g'(x)}{(g(x))^2},$$

but the proof is too involved so we will skip it.
However we are now in a position to differentiate many functions:

Example: If $f(x) = \dfrac{x^2+1}{x+\frac{1}{x}}$, calculate $f'(x)$.

Solution: The quotient rule gives

$$f'(x) = \frac{(x^2+1)'(x+\frac{1}{x}) - (x^2+1)(x+\frac{1}{x})'}{(x+\frac{1}{x})^2}.$$

We calculate

$(x^2+1)' = 2x$, and

$(x+\frac{1}{x})' = 1 + \left(\frac{1}{x}\right)' = 1 + \frac{-1}{x^2}$

Note: The derivative of $\frac{1}{x}$ is either an application of the quotient rule or the power rule for negative exponents.

Therefore

$$f'(x) = \frac{2x(x+\frac{1}{x}) - (x^2+1)(1-\frac{1}{x^2})}{(x+\frac{1}{x})^2}.$$

Example: Calculate the derivative of

$$f(x) = (x^2-3)(x^3+x+1)$$

using the product rule, and by multiplication of the factors first.

Solution: Using the product rule:

$$\begin{aligned} f'(x) &= (x^2-3)'(x^3+x+1) + (x^2-3)(x^3+x+1)' \\ &= 2x(x^3+x+1) + (x^2-3)(3x^2+1). \end{aligned}$$

Whereas if we multiplied/expanded the brackets:

$$\begin{aligned} f(x) &= x^5 + x^3 + x^2 - 3x^3 - 3x - 3 \\ &= x^5 - 2x^3 + x^2 - 3x - 3 \end{aligned}$$

and then

$$f'(x) = 5x^4 - 6x^2 + 2x - 3,$$

supposedly equal to our previous result.

Remark: In the end, the quotient rule will become redundant as it is a consequence of the product rule + "chain rule", which we will learn soon.

Example: Find the equation of the line tangent to

$$f(x) = \frac{1}{(x+1)^2} \quad \text{at} \quad x = 0.$$

Solution: We calculate

$$f'(x) = \frac{0 \cdot (x+1)^2 - (1)(x^2+2x+1)'}{(x+1)^4}$$

$$= \frac{\cancel{0(x^2+2x+1)} - (2x+2)}{(x+1)^4} = \frac{-2x+2}{(x+1)^4} = -2\frac{x+1}{(x+1)^4}$$

$$= \frac{-2}{(x+1)^3}$$

Therefore the slope of the tangent line is

$f'(0) = \frac{-2}{(0+1)^3} = -2$. The line passes through $(0, f(0))$
$= (0, 1)$.

So its equation is

$$y = -2(x-0) + 1$$
$$= -2x + 1.$$

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§2.4 The Chain Rule.

The chain rule tells us how to take derivatives of the composition of two functions.

Theorem: If $f(u)$ is differentiable at $u_0 = g(x_0)$ and g is differentiable at x_0, then the derivative of $(f \circ g)(x)$ at x_0 is given by:

$$(f \circ g)'(x) = f'(g(x_0)) \cdot g'(x_0)$$

or

$$\frac{d}{dx}(f(g(x))) = \frac{df}{du} \cdot \frac{dg}{dx} \cdot \quad (g(x) = u(x)).$$

"Proof": The proof of this rule is quite complicated, but it's easier if we assume that $g'(x)$ is never zero. If that's the case, then

$$(f \circ g)'(x_0) = \lim_{h \to 0} \frac{f(g(x_0+h)) - f(g(x_0))}{h}$$

$$\Rightarrow (f \circ g)'(x_0) \cdot \frac{1}{g'(x_0)} = \lim_{h \to 0} \frac{f(g(x_0+h)) - f(g(x_0))}{h} \cdot \frac{h}{g(x_0+h) - g(x_0)}$$

$$= \lim_{h \to 0} \frac{f(g(x_0+h)) - f(g(x_0))}{g(x_0+h) - g(x_0)}$$

this limit gives $\frac{1}{g'(x)}$

But this last limit, since g is continuous, is the same as

$$\lim_{g(x)\to g(x_0)} \frac{f(g(x)) - f(g(x_0))}{g(x) - g(x_0)} = f'(g(x_0)).$$

So we have

$$(f\circ g)'(x_0)\cdot \frac{1}{g'(x_0)} = f'(g(x_0)),$$ which is the chain rule.

<u>Note</u>: This is an <u>incomplete</u> proof since we need $g'(x_0) \neq 0$ for it to work.

<u>Example</u>: Calculate y' if $y = (3x+2)^{100}$

<u>Solution</u>: Here $y = f(g(x))$ where $f(x) = x^{100}$ and $g(x) = 3x+2$. So

$$f'(x) = 100x^{99} \Longrightarrow f'(g(x)) = 100(3x+2)^{99}$$

and

$$g'(x) = 3$$

Overall $(f\circ g)'(x) = 100(3x+2)^{99}\cdot 3.$

<u>Example</u>: Calculate the derivative of

$$y = \sqrt{\frac{x^2+1}{1-x^3}}.$$

Solution: Using Leibniz notation, we set $f(u) = \sqrt{u}$ and $u(x) = \dfrac{x^2+1}{1-x^3}$. Then

$$\frac{df}{du} = \frac{d}{du} u^{1/2} = \frac{1}{2} u^{-1/2}$$ (This uses the "generalized power rule" $\frac{d}{dx} x^p = p x^{p-1} \ \forall p$).

and

$$\frac{du}{dx} = \frac{2x(1-x^3) - (x^2+1)(-3x^2)}{(1-x^3)^2}.$$

Then

$$\frac{d}{dx} f(u(x)) = \frac{df}{du} \cdot \frac{du}{dx} = \frac{1}{2}\left(\frac{x^2+1}{1-x^3}\right)^{\frac{1}{2}} \cdot \frac{2x(1-x^3) - (x^2+1)(-3x^2)}{(1-x^3)^2}$$

Example: Calculate the derivative of

$$y = \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{x}}}}.$$

Solution: This is not two functions composed with one another, it is four functions. Unpack them as follows:

By the chain rule,

$$y' = \frac{1}{2}\left(1+\sqrt{1+\sqrt{1+\sqrt{x}}}\right)^{-1/2} \cdot \left(1+\sqrt{1+\sqrt{1+\sqrt{x}}}\right)'$$

and again by the chain rule,

$$\left(1+\sqrt{1+\sqrt{1+\sqrt{x}}}\right)' = \cancel{\frac{1}{2}\left(1+\sqrt{1+\sqrt{1+\sqrt{x}}}\right)^{-1/2}(}$$

$$= \left(\sqrt{1+\sqrt{1+\sqrt{x}}}\right)'$$

$$= \frac{1}{2}\left(1+\sqrt{1+\sqrt{x}}\right)^{-1/2} \cdot \left(1+\sqrt{1+\sqrt{x}}\right)'$$

again by the chain rule,

$$\left(1+\sqrt{1+\sqrt{x}}\right)' = \left(\sqrt{1+\sqrt{x}}\right)'$$

$$= \frac{1}{2}\left(1+\sqrt{x}\right)^{-1/2} \cdot \left(1+\sqrt{x}\right)'$$

$$= \frac{1}{2}\left(1+\sqrt{x}\right)^{-1/2} \cdot \frac{1}{2}x^{-1/2}.$$

Overall,

$$y' = \frac{1}{2}\left(1+\sqrt{1+\sqrt{1+\sqrt{x}}}\right)^{-1/2} \cdot \frac{1}{2}\left(1+\sqrt{1+\sqrt{x}}\right)^{-1/2} \cdot \frac{1}{2}\left(1+\sqrt{x}\right)^{-1/2} \cdot \frac{1}{2}x^{-1/2}.$$

<u>Remark</u>: In practice, you would omit all intermediate steps shown above, and cut straight to the answer.

<u>Example</u>: The quotient rule comes from the product rule and the chain rule!

Calculate the derivative of $\frac{f(x)}{g(x)}$ by writing it as $f(x) \cdot \frac{1}{g(x)}$. Then the product rule gives

$$\left(f(x)\frac{1}{g(x)}\right)' = f'(x) \cdot \frac{1}{g(x)} + f(x)\left(\frac{1}{g(x)}\right)'$$

and the chain rule gives

$$\left(\frac{1}{g(x)}\right)' = \frac{-1}{g(x)^2} \cdot g'(x) \quad \left(\text{think of composing } \frac{1}{x} \text{ with } g(x)\right).$$

So overall we have

$$\left(\frac{f(x)}{g(x)}\right)' = \left(f(x)\cdot\frac{1}{g(x)}\right)' = f'(x)\cdot\frac{1}{g(x)} + f(x)\cdot\frac{-g'(x)}{(g(x))^2}$$

$$= \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}.$$

<u>Example</u>: We can use the chain rule to differentiate $f(x) = |x|$, where it is differentiable. Write

$$f(x) = |x| = \sqrt{x^2}.$$

Define $g(u) = u^{1/2}$ and $u(x) = x^2$. Then $f = g \circ u$, so by the chain rule

$$\frac{df}{dx} = \frac{dg}{du}\cdot\frac{du}{dx},$$

here $\frac{dg}{du} = \frac{1}{2}u^{-1/2}$ and $\frac{du}{dx} = 2x$. So

$$\frac{df}{dx} = \frac{1}{2}(x^2)^{-1/2}\cdot 2x$$

$$= \frac{1}{\cancel{2}}\frac{1}{\sqrt{x^2}}\cdot\cancel{2}x = \frac{x}{|x|};$$ obviously this only works if $x \neq 0$.

So, we can also calculate the derivative of things like $f(x) = |g(x)|$.

<u>Example</u>: Compute the derivative of

$$f(x) = |x^2 + 2x - 1|$$

Solution: We calculate that since $f(x) = (g \circ u)(x)$ where $g(u) = |u|$ and $u(x) = x^2 + 2x - 1$, then:

$$\frac{dg}{du} = \frac{u}{|u|}, \quad u \neq 0 \quad \text{and} \quad \frac{du^*}{dx} = 2x + 2.$$

So

$$\frac{df}{dx} = \frac{dg}{du} \cdot \frac{du}{dx} = \frac{x^2+2x-1}{|x^2+2x-1|} \cdot 2x+2, \quad \text{with } x^2+2x-1 \neq 0.$$

So we also need to solve $x^2 + 2x - 1 = 0$, we get

$$x = -1 \pm \sqrt{2}.$$

So

$$\frac{df}{dx} = \frac{(x^2+2x-1)(2x+2)}{|x^2+2x-1|}, \quad \underbrace{x \neq -1 \pm \sqrt{2}}_{\text{This part is essential}}$$

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§2.5 Derivatives of trig functions.

Recall that we used the squeeze theorem to show that $\lim_{x \to 0} \frac{\sin x}{x} = 1$. With this we can show:

Example (The derivative rule for $\sin(x)$).

If $f(x) = \sin(x)$, show that $f'(x) = \cos(x)$.

Solution: By definition,

$$f'(x) = \lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sin\left(\left(x+\frac{h}{2}\right)+\frac{h}{2}\right) - \sin\left(\left(x+\frac{h}{2}\right)-\frac{h}{2}\right)}{h}$$

Now use the trig identity $\cos\alpha \sin\beta = \frac{\sin(\alpha+\beta) - \sin(\alpha-\beta)}{2}$

$$= \lim_{h \to 0} \frac{2\cos\left(x+\frac{h}{2}\right)\sin\left(\frac{h}{2}\right)}{h}$$

$$= \lim_{h \to 0} \cancel{2}\cos\left(x+\frac{h}{2}\right) \cdot \lim_{h \to 0} \frac{\sin\left(\frac{h}{2}\right)}{h/2}$$

$$= \cos(x) \cdot 1$$

here we use the fact that $\cos(x)$ is continuous for all x.

By a similar argument (using the limit $\lim_{h \to 0} \frac{\cos(h)-1}{h}$) we can show that $\boxed{\frac{d}{dx}\cos(x) = -\sin(x)}$

We can derive other trig derivatives from these two formulas and their relationship with sin/cos.

Example: Calculate $\frac{d}{dx}\tan x$.

Solution: Since $\tan(x) = \frac{\sin(x)}{\cos(x)}$, we can use the quotient rule and get:

$$\frac{d}{dx}\tan(x) = \frac{\left(\frac{d}{dx}\sin(x)\right)\cos(x) - \sin(x)\frac{d}{dx}\cos(x)}{\cos^2(x)}$$

$$= \frac{\cos^2 x - (\sin(x)(-\sin(x)))}{\cos^2(x)} \quad \left(\text{use } \cos^2 x + \sin^2 x = 1\right)$$

$$= \frac{1}{\cos^2(x)} = \sec^2(x)$$

Thus $\boxed{\frac{d}{dx}\tan x = \sec^2 x}$

Example: Find the derivative of $y = \sin(\cos(\tan(x)))$

Solution: Applying the chain rule to these nested functions, we get

$$y' = \cos(\cos(\tan(x))) \cdot -\sin(\tan(x)) \cdot \sec^2(x)$$

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There are also other trig formulas, all can be derived from the quotient or chain rule:

$$\frac{d}{dx}\sec x = \sec x \tan x, \quad \frac{d}{dx}\cot x = -\csc^2 x$$

$$\frac{d}{dx}\csc x = -\csc x \cot x.$$

Example: Find the equation of the line tangent to $y = \tan\left(\frac{\pi x}{4}\right)$ at the point $(1, 1)$.

Solution: We compute the slope from the derivative

$$\left.\frac{d}{dx}\left(\tan\left(\frac{\pi}{4}x\right)\right)\right|_{x=1} = \left.\sec^2\left(\frac{\pi}{4}x\right)\cdot\frac{\pi}{4}\right|_{x=1}$$

comes from chain rule

$$= \frac{\pi}{4}\left(\sqrt{2}\right)^2 = \frac{\pi}{2}.$$

So from the point-slope formula, the tangent line is

$$y = \frac{\pi}{2}(x-1) + 1.$$

Example (#48 §2.5)

Show that the graphs of $y = \tan x$ and $y = \cot x$ never have horizontal ~~tangents~~ lines.

Solution: A horizontal tangent line means the derivative is zero. So we must show the derivatives of the given functions can never be zero.

For $y = \tan x$, since $y' = \sec^2(x) = \frac{1}{\cos^2(x)}$ we know $y' = 0$ is impossible since $\frac{1}{\cos^2(x)}$ is not possible ($\frac{1}{y} \neq 0$ for any y).

Similarly if $y = \cot x$ then $y' = -\csc^2 x = \frac{-1}{\sin^2 x}$, which can never be zero.

Example: From the trig identity $\sin(2x) = 2\sin x \cos x$, deduce the identity

$$\cos 2x = \cos^2 x - \sin^2 x.$$

Solution: By differentiating both sides of the equation, we arrive at

$$\cos 2x \cdot 2 = 2\left((\sin x)'\cos x + \sin x(\cos x)'\right)$$

$$= 2\left(\cos^2 x + \sin(x)(-\sin(x))\right)$$

$$\Rightarrow \cos 2x = \cos^2 x - \sin^2 x.$$ If we differentiate again, we arrive at the original $\cos 2x = \cos^2 x - \sin^2 x$.

In fact, many trig identities "come in pairs" in this way.

Example: Deduce the angle sum/difference formulas for cosine from

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y.$$

Solution: If we differentiate, treating x as the variable and y as a constant, we find:

$$\cos(x \pm y) = \cos y \cos x \pm (-\sin x)\sin y$$
$$= \cos y \cos x \mp \sin x \sin y.$$

Note that here $\frac{d}{dx}(\cos y \cos x) = \cos y \frac{d}{dx}\cos(x)$, since y is treated as a constant.

Example: If $f(x) = \cos^3(x)$, solve $f'(x) = 0$ for all solutions in $[0, 2\pi]$.

Solution: From the chain rule

$$f'(x) = 3\cos^2(x) \cdot (\cos(x))'$$
$$= -3\cos^2(x)\sin(x)$$

So if $f'(x) = -3\cos^2(x)\sin(x) = 0$, then either $\sin(x) = 0$ or $\cos(x) = 0$.

- The solutions to $\sin(x) = 0$ in $[0, 2\pi]$ are $x = 0, \pi, 2\pi$
- The solutions to $\cos(x) = 0$ in $[0, 2\pi]$ are $x = \frac{\pi}{2}, \frac{3\pi}{2}$.

Thus $f'(x) = 0$ at the points

$x = 0, \pi/2, \pi, \frac{3\pi}{2}, 2\pi$ in $[0, 2\pi]$.