MATH 1230

Lecture 2

§P1 Inequalities and real numbers, absolute values

We will use inequalities often. To "solve" an inequality means to isolate the variable in the equation. Answers are stated as inequalities or intervals.

Example:

Solve $-3(4-x) \leq 12$.

Solution:

$$-3(4-x) \leq 12$$

$$4-x \geq -4 \quad \text{(multiply both sides by } -\tfrac{1}{3}\text{)}$$

$$-x \geq -8$$

$$x \leq 8 \quad \text{(multiply by } -1\text{)}$$

Our answer is either $x \leq 8$ or $x \in (-\infty, 8]$

This symbol means "is a member of" or simply "is in".

Example: Solve $\dfrac{2}{x-1} - 5 \geq 0$.

Solution:

$$\frac{2}{x-1} - 5 \geq 0$$

$$\Rightarrow \frac{2-5(x-1)}{x-1} \geq 0 \Rightarrow \frac{7-5x}{x-1} \geq 0.$$

To determine the sign of a fraction/product, we consider the signs of the factors involved.

So $7-5x=0$ when $x=\frac{7}{5}$

$x-1=0$ when $x=1$.

These are the places where the factors change sign.

function		1		$\frac{7}{5}$	
$7-5x$	$+$		$+$		$-$
$x-1$	$-$		$+$		$+$
$\frac{7-5x}{x-1}$	$-$		$+$		$-$

So $\frac{7-5x}{x-1}$ is positive (>0) in $(1, \frac{7}{5})$. Testing the endpoints, we get $\frac{7-5x}{x-1}$ undefined @ $x=1$

$$\frac{7-5x}{x-1}=0 \text{ @ } x=\frac{7}{5}.$$

So the solution is $x \in (1, \frac{7}{5}]$ (since we want ≥ 0).

<u>Example</u>: Solve $\frac{2}{x-1} < \frac{1}{2x}$.

<u>Solution</u>: Do <u>not</u> cross-multiply, giving

$$2(2x) < 1 \cdot (x-1).$$

This would be <u>wrong</u>, because x or $x-1$ may be negative, meaning we would have to replace $<$ with $>$. Instead:

$$\frac{2}{x-1} - \frac{1}{2x} < 0$$

$$\Rightarrow \frac{4x-(x-1)}{(x-1)(2x)} < 0 \Rightarrow \frac{3x+1}{(x-1)(2x)} < 0.$$

Now we make a table, using "break points" of $x = -\frac{1}{3}, 0, 1$:

function		$-\frac{1}{3}$		0		1		$\mathbb{R}$
$3x+1$	$-$		$+$		$+$		$+$	
$x-1$	$-$		$-$		$-$		$+$	
$2x$	$-$		$-$		$+$		$+$	
$\frac{3x+1}{(x-1)(2x)}$	$-$		$+$		$-$		$+$	

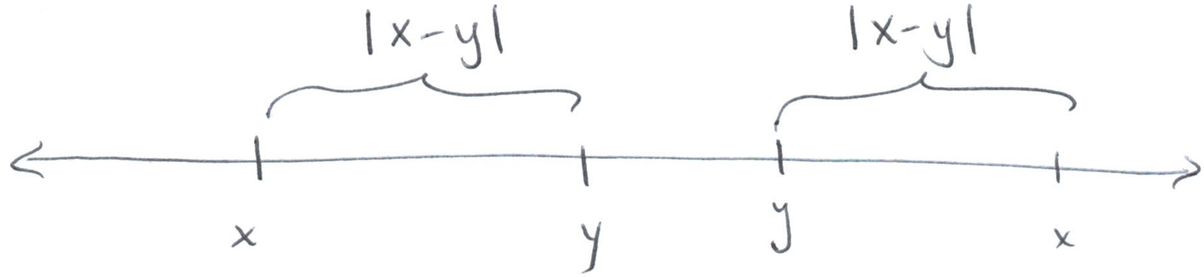
So the solution is $x \in (-\infty, -\frac{1}{3}) \cup (0, 1)$

union.

Absolute values will play an important part in our work.

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Geometrically, $|x|$ is the distance from x to 0, and $|x-y|$ is the distance from the number x to the number y:



Absolute value has 3 important properties:

(i) $|-x| = |x|$

(ii) $|x||y| = |xy|$

(iii) $|x \pm y| \leq |x| + |y|$ (called the triangle inequality).

We use the properties above to solve inequalities that involve absolute values.

<u>Example</u>: Solve $|3x-4| < 2$.

<u>Solution</u>: Since $|a| < b$ means $-b < a$ and $a < b$, $|3x-4| < 2$ means

$$-2 < 3x-4 \quad \text{and} \quad 3x-4 < 2.$$

or, more simply,

$$-2 < 3x-4 < 2.$$

$$\Rightarrow \quad 2 < 3x < 6$$

$$\Rightarrow \quad \tfrac{2}{3} < x < 2.$$

So $x \in \left(\frac{2}{3}, 2\right)$ is the solution.

<u>Example</u>: Solve $|3x-4| < \varepsilon$, where $\varepsilon > 0$ is some positive number.

<u>Solution</u>: As before,

$$-\varepsilon < 3x-4 < \varepsilon$$

$$\Rightarrow 4-\varepsilon < 3x < \varepsilon+4$$

$$\Rightarrow \frac{4-\varepsilon}{3} < x < \frac{\varepsilon+4}{3}$$

$$\Rightarrow x \in \left(\frac{4-\varepsilon}{3}, \frac{\varepsilon+4}{3}\right).$$

Other times, we will want to start with one inequality, and derive another from it.

<u>Example</u>: Show that if $-2 \le x \le 2$, then $|x^2+2x+2| \le 10$.

<u>Solution</u>: Since we see absolute value bars around a sum, we should think of applying the triangle inequality: $|a+b| \le |a| + |b|$.

So here:
$$|\underbrace{x^2+2x}_{a}+\underbrace{2}_{b}| \le \underbrace{|x^2+2x|}_{a} + \underbrace{|2|}_{b}$$

$$\le |x^2|+|2x|+|2|.$$

Our given inequality, $-2 \le x \le 2$, is the same as $|x| \le 2$. This also gives $|x|\cdot|x| = |x^2| < 4$, upon squaring both sides. So then

$$\begin{aligned} |x^2+2x+2| &\le |x^2| + 2|x| + |2| \\ &\le 4 + 2\cdot 2 + 2 \\ &= 10, \text{ so we are done.} \end{aligned}$$

But why is the triangle inequality true?

Example: Show that $|a+b| \leq |a| + |b|$ for all $a, b \in \mathbb{R}$.

Proof: For an arbitrary real number a, we always have $a \leq |a|$, and $-|a| \leq a$. The same is true for b: we always have $b \leq |b|$ and $-|b| \leq b$.

Therefore $a+b \leq |a| + |b|$

and $-|a| - |b| \leq a+b$.

Together, $-|a| - |b| \leq a+b \leq |a| + |b|$, or $|a+b| \leq |a| + |b|$.

Example: If $a \leq b$, show $a \leq \frac{a+b}{2} \leq b$.

Proof: First we note:

$$a = \tfrac{1}{2}(a+a) \leq \tfrac{1}{2}(a+b), \text{ since } a \leq b.$$

similarly, $\frac{1}{2}(b+b) = b$

and $\frac{1}{2}(a+b) \leq \frac{1}{2}(b+b) = b$.

Overall, $a \leq \frac{1}{2}(a+b) \leq b$.

Example: We may also encounter quadratic inequalities. Solve

$$x^2 - 4x + 3 \leq 0.$$

Solution: Since $x^2 - 4x + 3 = (x-3)(x-1)$, we need these factors to have opposite signs to produce a negative

That is, we need either:

(i) $x-3 \geq 0$ and $x-1 \leq 0$ or

(ii) $x-3 \leq 0$ and $x-1 \geq 0$.

Either $x \geq 3$ and $x \leq 1$ <u>or</u> $x \leq 3$ and $x \geq 1$.

Either

<u>or</u>

no solution!

solution!

So $x \in [1, 3]$.

<u>Remark</u>: It is also possible to understand absolute value as <u>distance</u>; so $|x-a|$ can be interpreted as "the distance from x to a".
c.f. Example 8 in §P1 and problems 41 & 42.

<u>Asst 1</u>:

P1 42

P4 6

P5 27

P7 4, 24

§1.2 20, 24.

MATH 1230.

§ P4 Functions.

We use functions to describe how one quantity depends on another. E.g

$$A(r) = \pi r^2$$

deseribes they dependency between Area and radius of a circle. A "generic" function, when we have no particular formula in mind, will be called $f(x)$, or $y = f(x)$. Terminology:

- x is the independent variable
- y is the dependent variable.
- the domain is the set of all "inputs"
- the range is the set of outputs.

Convention: Sometimes we will give a formula for a function without specifying the domain explicitly. In this case, we assume the domain of the function, $f(x)$, say, is all real numbers x for which $f(x)$ is defined.

Example: Let $f(x) = \sqrt{x}$. What is the domain of $f(x)$?

Solution: First, a remark about the square root function: $\sqrt{x}$ is always a positive number. For example, $\sqrt{4} = 2$, not ± 2.

The domain of $f(x)$ is all numbers which have a square root, namely $[0, \infty)$.

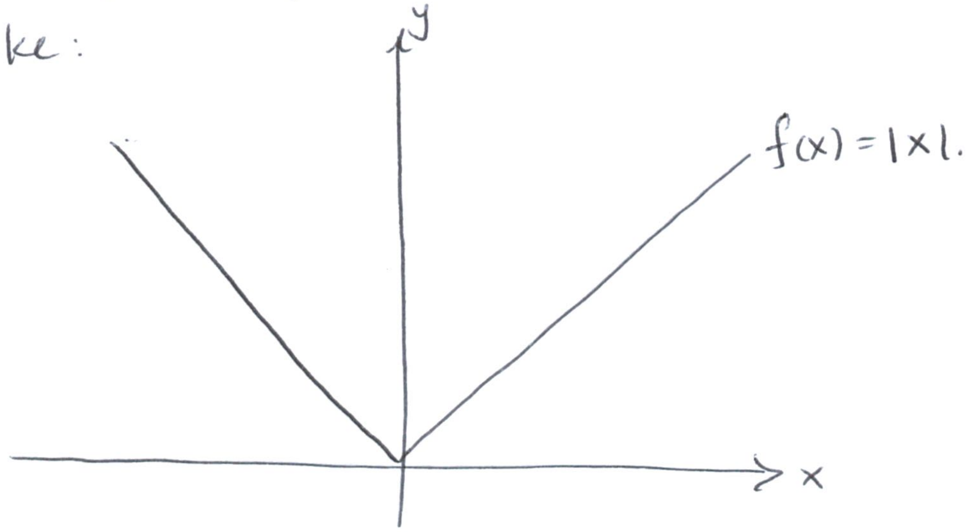
Example: What is the domain of

$$f(x) = \sqrt{\frac{7-5x}{x-1}} \; ?$$

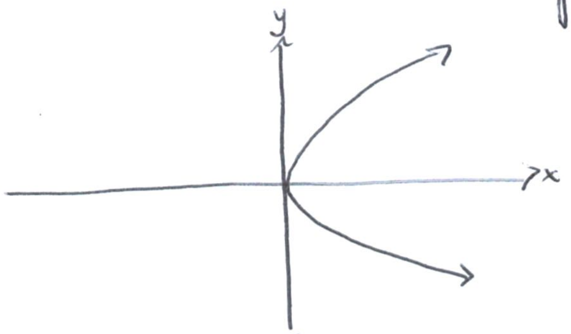
Solution: Last day we calculated that $\frac{7-5x}{x-1}$ is positive in $(1, 7/5)$, undefined at $x=1$ and 0 at $x = 7/5$. Therefore the domain is $(1, 7/5]$.

Graphing basics

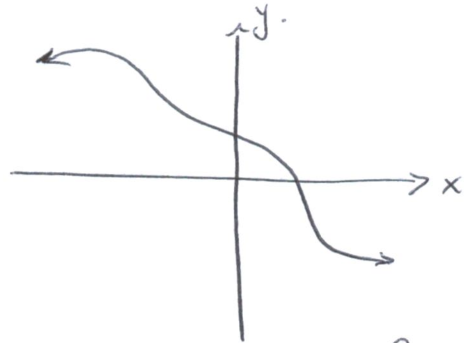
Given a function $f(x)$, we graph it by plotting all points of the form $(x, f(x))$. This results in things like:



Not every curve comes from a function in this way:
If there is a vertical line that intersects a curve twice, that curve is <u>not</u> the graph of a function. Otherwise, it is the graph of a function:



not a function



a function

You should be familiar with the graphs of several functions already:

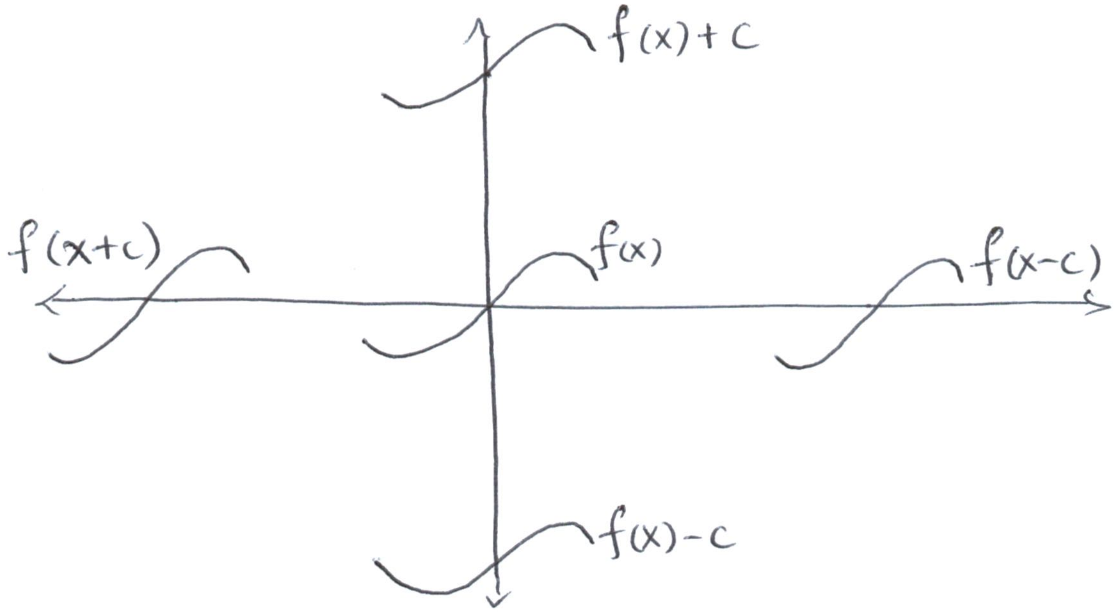
- lines
- quadratics
- square roots
- $f(x) = x^3$
- $f(x) = \frac{1}{x}$
- $f(x) = \frac{1}{x^2}$

For a complete list, see page 27.

We can make new function from old ones by doing geometric operations on their graphs which have corresponding algebraic operations on their equations.

<u>Shifting</u>:

Suppose $c > 0$. We can shift horizontally or vertically as follows:



<u>Stretching</u>: If $c>0$,

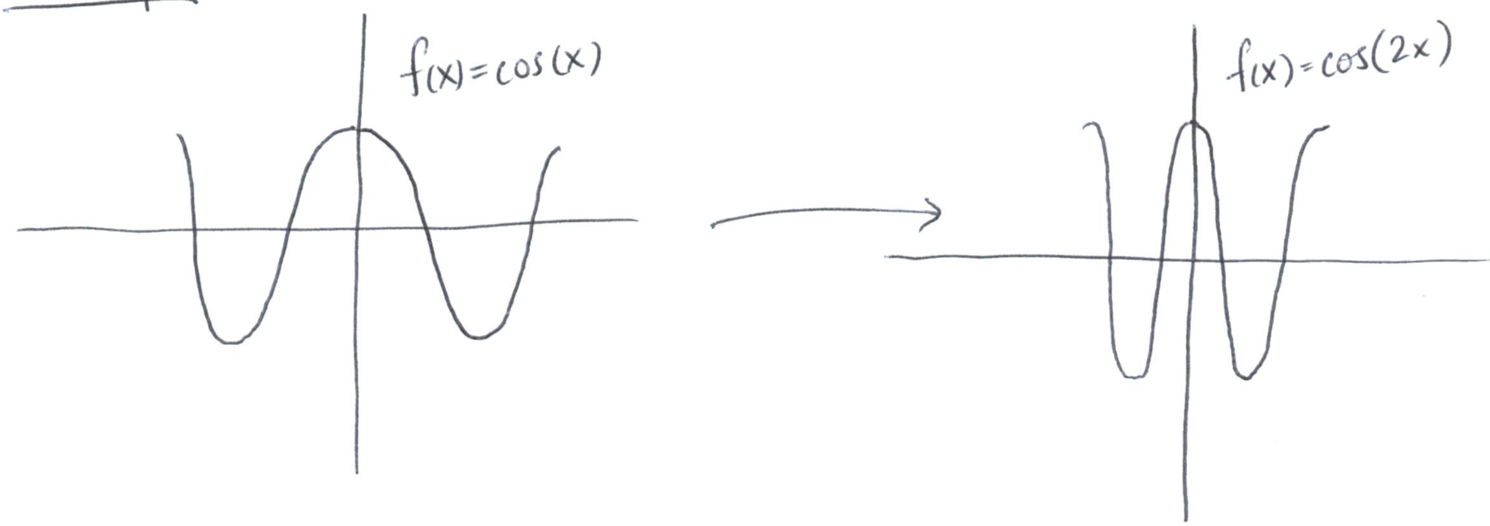
(i) $f(x) \longmapsto cf(x)$ stretch vertically by a factor of c

(ii) $f(x) \longmapsto \frac{1}{c}f(x)$ compress " " " " " c.

(iii) $f(x) \longmapsto f(cx)$ compress horizontally

(iv) $f(x) \longmapsto f\left(\frac{1}{c}x\right)$ stretch horizontally

<u>Example</u>:

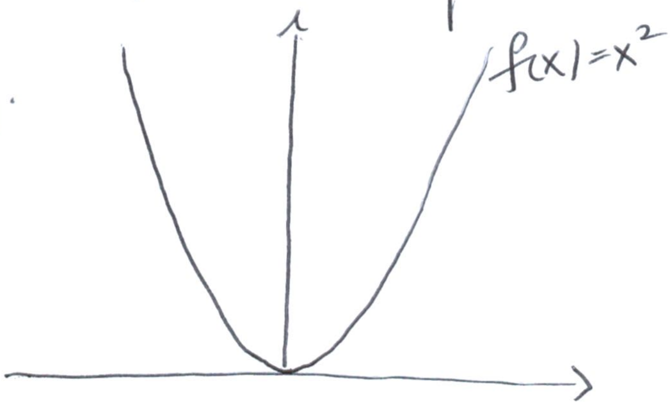


Even and odd functions:

A function $f(x)$ is <u>even</u> if $f(-x) = f(x)$ for all ~~$x \in \mathbb{R}$~~ x in the domain.

Graphically: The function is symmetric about the y-axis:

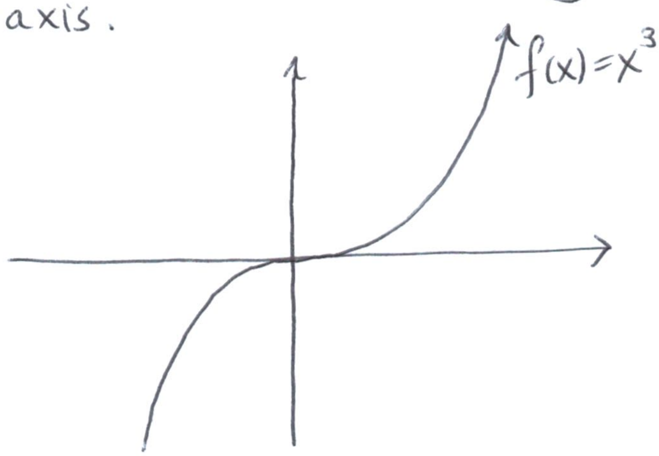
E.g.



$$f(-x) = (-x)^2 = x^2$$

A function is <u>odd</u> if $f(-x) = -f(x)$ for all x in the domain. In this case, the graph of $f(x)$ remains the same after reflecting in the y-axis, and then the x-axis.

E.g.



$$f(-x) = (-x)^3 = -x^3 = -f(x)$$

<u>Question</u>: Is there a function that is both even <u>and</u> odd?

<u>Answer</u>: Yes, $f(x)=0$ is both even and odd:

$f(-x) = 0 = f(x)$ (it's even)

$f(-x) = 0 = -f(x)$ (it's odd).

<u>Claim</u>: $f(x)=0$ is the only function that is both even and odd.

<u>Proof</u>: Suppose $g(x)$ is even and odd.

Then $g(-x) = g(x)$ (even)

and $g(-x) = -g(x)$ (odd)

so $g(x) = -g(x)$

$\Rightarrow 2g(x)=0$, so $g(x)$ must be 0 for all x.

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Last, recall that we can add, subtract, multiply and divide functions:

$(f+g)(x) = f(x)+g(x)$

$(f-g)(x) = f(x)-g(x)$

$(fg)(x) = f(x)g(x)$

$(f/g)(x) = f(x)/g(x)$, as long as $g(x) \neq 0$.

We can also <u>compose</u> functions by plugging one into the othere:

$$\underbrace{(f \circ g)(x)}_{\text{composition of } f \text{ with } g} = f(\underbrace{g(x)}_{\text{do } g \text{ first, then do } f})$$

Example: Suppose $f(x) = \frac{x+1}{x-2}$ and $g(x) = \sqrt{x+1}$.
What is the domain of $g(f(x))$? $(g \circ f)$

Solution: First we calculate.

$$g(f(x)) = g\left(\frac{x+1}{x-2}\right) = \sqrt{\frac{x+1}{x-2} + 1}$$

$$= \sqrt{\frac{x+1+x-2}{x-2}}$$

$$= \sqrt{\frac{2x-1}{x-2}}$$

The domain of $g \circ f$ is restricted in the following two ways:

(i) The quantity under the square root cannot be negative

(ii) We mustn't divide by zero.

So, (i) gives us:

$\frac{2x-1}{x-2} \geq 0$. We make a quick table:

function		$x=\frac{1}{2}$		$x=2$	
$2x-1$	$-$		$+$		$+$
$x-2$	$-$		$-$		$+$
$\frac{2x-1}{x-2}$	$+$		$-$		$+$

And $\frac{2x-1}{x-2} = 0$ when $x = \frac{1}{2}$, and is undefined when $x = 2$. So we find $(-\infty, \frac{1}{2}] \cup (2, \infty)$ as the intervals of x-values satisfying (i).

To satisfy (ii), we need only $x \neq 2$.
So the domain is $(-\infty, \frac{1}{2}] \cup (2, \infty)$.

A function is called <u>piecewise defined</u> if it uses different formulas on different parts of its domain, e.g.

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$$

However we might encounter something much more complex:

<u>Example</u>: Sketch a graph of the function

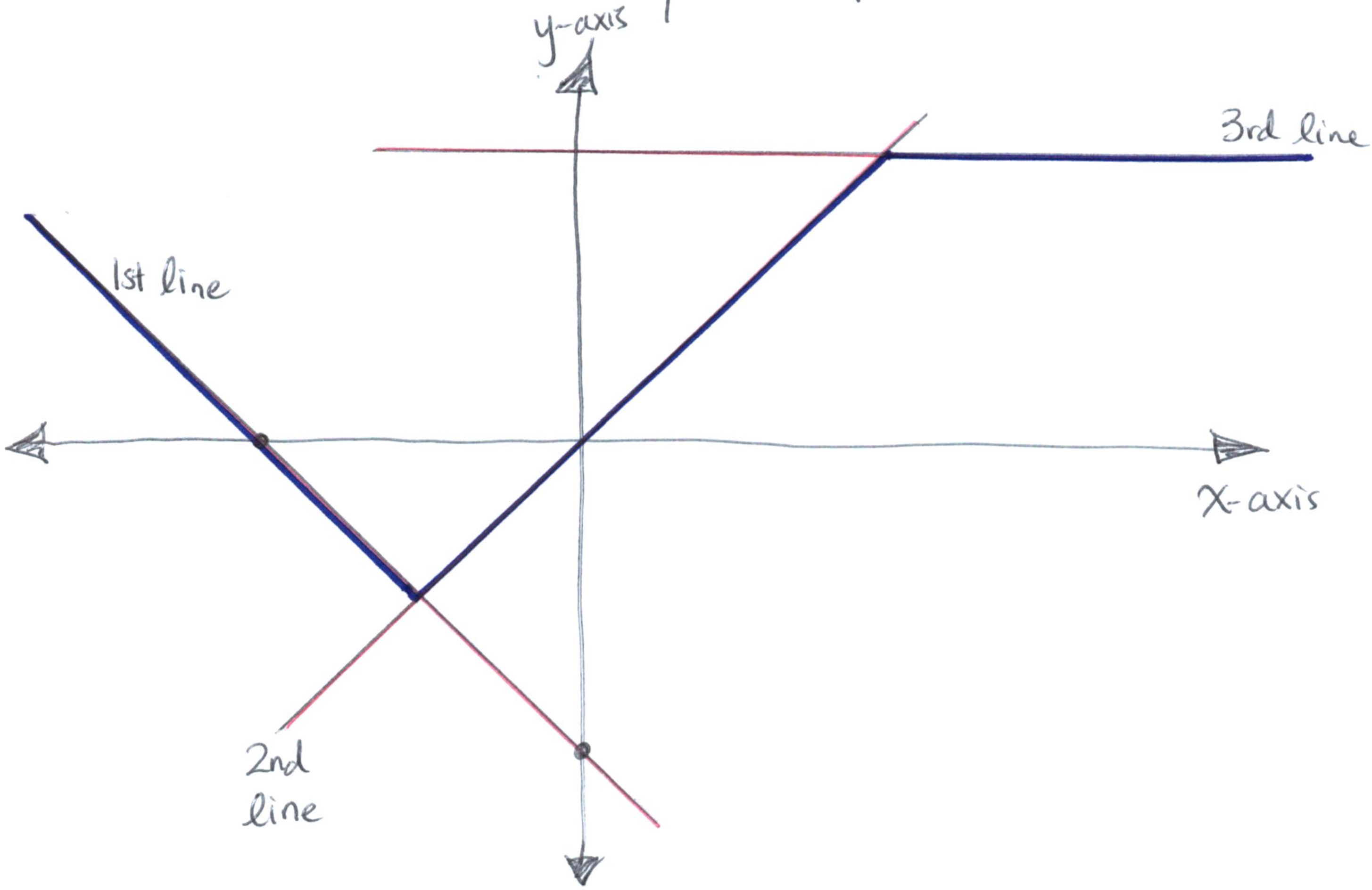
$$f(x) = \begin{cases} -(x+2) & \text{if } x < -1 \\ x & \text{if } -1 \leq x \leq 2 \\ 2 & \text{if } x > 2. \end{cases}$$

<u>Solution</u>: We know how to graph straight lines, and this function consists of 3 of them.

<u>First line</u>: Slope -1, y-intercept -2
x intercept -2

<u>Second line</u>: Slope 1, x intercept 0
y intercept 0

Third line: Slope 0, no x intercept
y intercept 2.



The three lines are plotted above, but because of the domain restrictions we only take the portions coloured blue. So f(x) is the graph in blue above.

<u>Example</u>: Prove that the composition of two even functions is even. Prove that the composition of an even and odd function is even.

<u>Proof</u>: Suppose $f(x)$ and $g(x)$ are both even. Then

$$(f\circ g)(-x) = f(g(-x)) = f(g(x)) = (f\circ g)(x)$$

because g is even.

so $f\circ g$ is even.

Now suppose $f(x)$ is even and $g(x)$ is odd, then:

$$(f\circ g)(-x) = f(g(-x)) = f(-g(x)) = f(g(x)) = (f\circ g)(x)$$

because g is odd   because f is even.

Therefore $(f\circ g)(x)$ is even. Last, if we do the functions in the other order:

$$(g\circ f)(-x) = g(f(-x)) = g(f(x)) = (g\circ f)(x)$$

because f is even

So $g\circ f$ is even, too.

<u>Extra</u>: Can you prove that the composition of an odd function and an odd function is odd?

<u>Remark</u>: Note that even/odd functions and composition behave the same as even/odd numbers and multiplication.

There are also some common piecewise-defined functions that you will encounter in your mathematical career:

<u>Example</u>: The Heaviside or unit step function:

$$H(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0. \end{cases}$$

<u>Example</u>: The sign function, written $\operatorname{sgn}(x)$,

$$\operatorname{sgn}(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \\ \text{undefined} & \text{if } x = 0. \end{cases}$$

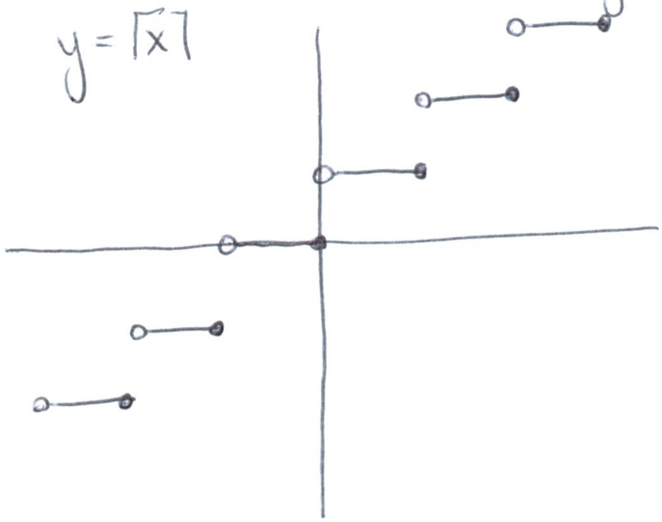
This function tells you if a quantity is positive or negative.

<u>Example</u>: The ceiling and floor functions

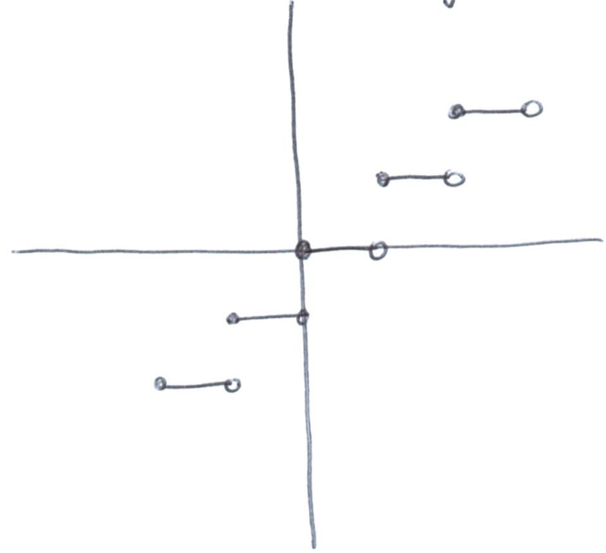
$$y = \lceil x \rceil \quad \text{and} \quad y = \lfloor x \rfloor$$

also called the "greatest integer" and "least integer" functions. Here is a graph of each:

$y = \lceil x \rceil$



$y = \lfloor x \rfloor$



The formulas are:

$\lceil x \rceil = n$ on each interval $(n-1, n]$, n an integer

and

$\lfloor x \rfloor = n$ on each interval $[n, n+1)$, n an integer.

These functions return the least integer greater than x and the greatest integer less than x respectively.