MATH 1230

Logarithmic differentiation and review

Logarithmic differentiation allows for the differentiation of complicated expressions—particularly expressions with complicated powers— by first taking ln of both sides.

Example: If $y = x^{\sqrt{x}}$, what is y'?

Solution: Take ln of both sides. Then

$$\ln(y) = \ln(x^{\sqrt{x}}) = \sqrt{x}\,\ln(x)$$

Now differentiate (implicitly!)

$$\frac{d}{dx}\ln(y) = \frac{d}{dx}\sqrt{x}\,\ln(x)$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \left[\frac{d}{dx}\sqrt{x}\right]\ln(x) + \sqrt{x}\,\frac{d}{dx}\ln(x)$$

$$= \frac{1}{2}x^{-1/2}\ln(x) + \sqrt{x}\cdot\frac{1}{x}$$

So $$\frac{dy}{dx} = y\left(\frac{1}{2}\sqrt{x}\,\ln(x) + x^{-1/2}\right)$$

$$= x^{\sqrt{x}}\left(\frac{1}{2}\sqrt{x}\,\ln(x) + \frac{1}{\sqrt{x}}\right).$$

Final exam: St John's Room 118

Example: If

$f(x) = (2x+3)^{150}(x^2+1)^{75}\left(x+\frac{1}{x}\right)^{12}(x-1)^{13}$, find $f'(x)$.

Solution: We could use traditional ~~trig~~ differentiation rules, which would require many applications of the chain and product rule. Alternatively:

$$y = (2x+3)^{150}(x^2+1)^{75}\left(x+\frac{1}{x}\right)^{12}(x-1)^{13}$$

$$\Rightarrow \ln(y) = 150\ln(2x+3) + 75\ln(x^2+1) + 12\ln\left(x+\frac{1}{x}\right) + 13\ln(x-1)$$

$$\Rightarrow \frac{1}{y}y' = \frac{150}{2x+3}\cdot 2 + \frac{75}{x^2+1}\cdot 2x + \frac{12}{x+\frac{1}{x}}\left(1-\frac{1}{x^2}\right) + \frac{13}{x-1}$$

$$\Rightarrow y' = (2x+3)^{150}(x^2+1)^{75}\left(x+\frac{1}{x}\right)^{12}(x-1)^{13}\cdot\left(\frac{150}{2x+3}\cdot 2 + \frac{75}{x^2+1}\cdot 2x + \frac{12}{x+\frac{1}{x}}\left(1-\frac{1}{x^2}\right) + \frac{13}{x-1}\right).$$

That's implicit differentiation.

THE UNIVERSITY OF MANITOBA
DEPARTMENT OF MATHEMATICS

MATH 1230 MIDTERM TEST
17 December 2015   1:30–3:30 PM
Examiner: D. Krepski

Name (print): ______________________________

Signature: ______________________________

ID number: ______________________________

Please circle your Tutorial/Lab Section below:

B01 Fridays 1:30
B02 Fridays 8:30
B03 Fridays 9:30

INSTRUCTIONS:

1. This exam is 11 pages long. It is printed double-sided. There are 9 questions, including one bonus question. It is your responsibility to check that you have a complete exam booklet.
2. All questions must be answered in the space provided. Indicate your responses clearly, showing all of your work with complete explanations and full justification.
3. Do **not** unstaple the exam booklet.
4. Should you need extra space, a blank page is provided at the end of the booklet. Please clearly indicate if your response continues on this blank page (e.g. "continued on p. 11") to ensure your complete response is examined.
5. **No aids of any kind are permitted: electronic devices, notes, abacuses, etc. are not permitted.**

Question:	1	2	3	4	5	6	7	8	BONUS	Total
Marks:	15	14	4	8	6	8	18	9	0	82
Score:										

15 1. **MULTIPLE-CHOICE QUESTIONS**

Each of the following multiple-choice questions has exactly ONE correct answer. Clearly indicate your answer to each question by circling your response.

Marking scheme: 2.5 marks for selecting the correct choice; 0 marks for selecting a wrong choice, or not selecting a choice, or selecting more than one choice.

(a) Find $\lim_{x\to 2} \dfrac{x^2+x-6}{x^2-4}$.

A. 0
B. does not exist
C. 5/4
D. −1/4
E. 3/2

$$\lim_{x\to 2} \frac{x^2+x-6}{x^2-4} = \lim_{x\to 2} \frac{(x+3)(x-2)}{(x+2)(x-2)} = \lim_{x\to 2} \frac{x+3}{x+2} = \frac{2+3}{2+2} = \frac{5}{4}.$$

(b) Find $\lim_{x\to -\infty} \dfrac{2x^3+3}{7-3x^2-9x^3}$.

A. 2/9
B. 3/7
C. −2/9
D. $-\infty$
E. ∞

$$\lim_{x\to -\infty} \frac{2x^3+3}{7-3x^2-9x^3} = \lim_{x\to -\infty} \frac{2+3/x^3}{7/x^3-3/x-9} = \frac{-2}{9}$$

$\frac{0}{0}$ indeterminate form

(c) Find $\lim_{x\to 0} \dfrac{\sin(2x)\cos(5x)}{\cos(3x)\sin(4x)}$.

$$\overset{H.R.}{=} \lim_{x\to 0} \frac{\overset{1}{\cos(2x)}\cdot 2\cdot \overset{1}{\cos(5x)} - \cancel{\sin(2x)}\cdot 5\cdot \overset{0}{\sin(5x)}}{-3\cancel{\sin(3x)}\underset{0}{\sin(4x)} + \underset{1}{\cos(3x)}\cdot 4\cdot \underset{1}{\cos(4x)}} = \frac{2}{4} = \frac{1}{2}.$$

A. 2/3
B. 5/3
C. 1/2
D. 5/4
E. 0

(d) If $xy^3 + y - x = 23$, find the equation of the tangent line at the point $P(3, 2)$.

A. $y - 2 = \frac{4}{25}(x-3)$
B. $y - 2 = \frac{3}{38}(x-3)$
C. $y - 2 = \frac{5}{37}(x-3)$
D. $y - 2 = -\frac{6}{35}(x-3)$
E. $y - 2 = -\frac{7}{37}(x-3)$

Implicit diff:

$(y^3 + x3y^2 \cdot y') + y' - 1 = 0$

Then set $x=3$, $y=2$, and get

$2^3 + 3\cdot 3\cdot 2^2 \cdot y' + y' - 1 = 0$

$\Rightarrow \quad 37y' = \cancel{8} -7$

$y' = \frac{-7}{37}.$

The point-slope formula gives

$y - 2 = \frac{-7}{37}(x-3).$

(e) The product of two positive numbers is 100. What is the smallest possible value for their sum?

A. 18
B. 19
C. 20
D. 21
E. 22

Call the numbers x and y. Then $xy = 100$, minimize $x+y$ subject to this constraint.

$\Rightarrow y = \frac{100}{x}$, so we want to minimize

Sum $= x + y = x + \frac{100}{x}$, where $x > 0$. Differentiate:

$(\text{Sum})' = 1 - \frac{100}{x^2}$. Then $1 - \frac{100}{x^2} = 0 \Rightarrow \frac{1}{x^2} = \frac{1}{100} \Rightarrow x = 10.$

Then $y = 10$ and so $x + y = 20$.

(f) The n-th order Taylor polynomial of a function f about $x = 0$ is

$$P_n(x) = 1 + 2x + 3x^2 + 4x^3 + \cdots + (n+1)x^n.$$

What is $f^{(1230)}(0)$?

A. 1230!
B. 1/1230!
C. 1231/1230!
D. 1231!
E. 1231

$f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n.$

So $\frac{f^{(n)}(0)}{n!}x^n = (n+1)x^n$

$\Rightarrow f^{(n)}(0) = n!(n+1)$

$= 1230! \cdot (1231) = 1231!$

2. Differentiate the following functions. (You do not need to simplify your answer.)

$\boxed{4}$ (a) $f(x) = (\ln x + \sin x)(\tan x + 1)$

$$f'(x) = \left(\ln(x) + \sin x\right)'\left(\tan x + 1\right) + \left(\ln(x) + \sin(x)\right)\left(\tan x + 1\right)'$$

$$= \left(\frac{1}{x} + \cos(x)\right)\left(\tan x + 1\right) + \left(\ln(x) + \sin(x)\right)\left(\sec^2 x\right)$$

$\boxed{5}$ (b) $f(x) = \dfrac{e^x \sin(2x)}{\sqrt{1-3x^2}}$

$$f'(x) = \frac{\left(e^x \sin(2x)\right)'\sqrt{1-3x^2} - e^x \sin(2x)\left(\sqrt{1-3x^2}\right)'}{1-3x^2}$$

$$= \frac{\left(e^x \sin(2x) + 2e^x \cos(2x)\right)\sqrt{1-3x^2} - e^x \sin(2x)\frac{1}{2}\left(1-3x^2\right)^{-\frac{1}{2}} \cdot -6x}{1-3x^2}.$$

$\boxed{5}$ (c) $f(x) = x^{(e^x)}$

If $y = x^{e^x} \Rightarrow \ln(y) = \ln(x^{e^x}) = e^x \ln(x)$.

$$\Rightarrow \frac{1}{y} y' = e^x \ln(x) + e^x \cdot \frac{1}{x}$$

$$\Rightarrow y' = x^{e^x}\left(e^x \ln(x) + e^x \cdot \frac{1}{x}\right).$$

$\boxed{4}$ 3. Circle ONE of the following theorems, and give the full precise statement of that theorem:

- The Intermediate Value Theorem
- The Mean Value Theorem

IVT: If $f(x)$ is continuous on $[a,b]$ and if s is a number between $f(a)$ and $f(b)$, then there exists c in $[a,b]$ with $f(c)=s$.

MVT: Suppose that $f(x)$ is continuous on $[a,b]$ and differentiable on (a,b). Then there exists c in (a,b) such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$

$\boxed{3}$ 4. (a) State the precise definition of $\lim_{x\to\infty} f(x) = L$.

$\lim_{x\to\infty} f(x) = L$ if for every $\varepsilon > 0$ there exists a real number M such that $x > M$ implies $|f(x) - L| < \varepsilon$.

$\boxed{5}$ (b) Using the precise definition of limit, show that $\lim_{x\to\infty} \frac{1}{\sqrt{x-1}} = 0$.

Let $\varepsilon > 0$ be given. Then to find M such that $x > M$ implies $\frac{1}{\sqrt{x-1}} < \varepsilon$, we solve:

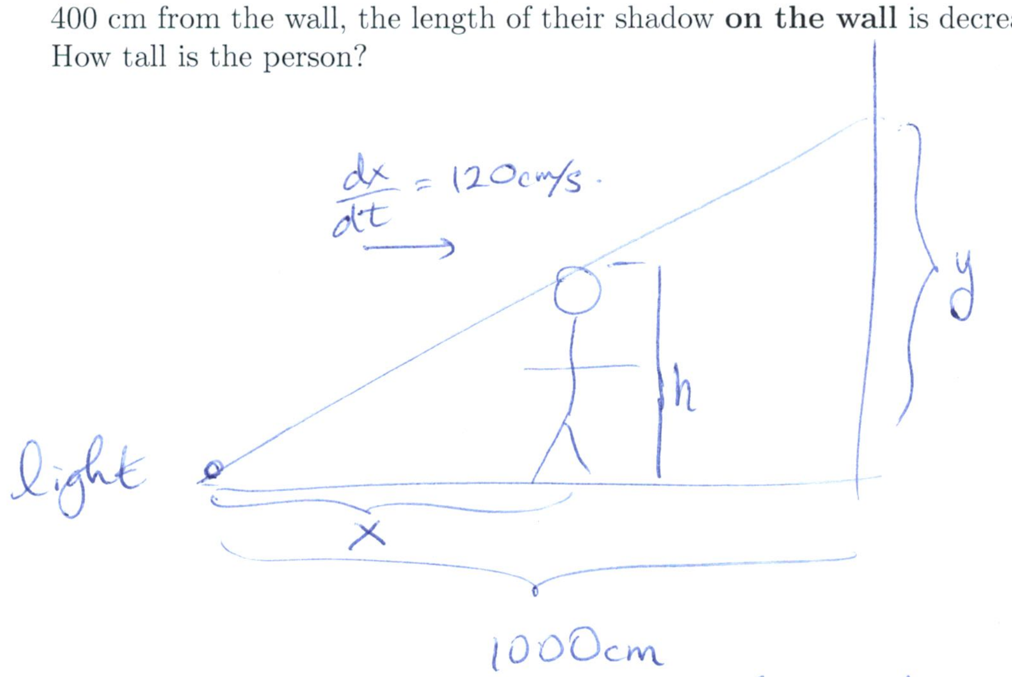
$$\left|\frac{1}{\sqrt{x-1}}\right| < \varepsilon \iff \sqrt{x-1} > \varepsilon$$

$$\iff x-1 > \varepsilon^2$$

$$\iff x > \varepsilon^2 + 1.$$

So choose $M = \varepsilon^2 + 1$. Then from the above calculation, $x > M$ implies $\left|\frac{1}{\sqrt{x-1}}\right| < \varepsilon$, so $\lim_{x\to\infty} \frac{1}{\sqrt{x-1}} = 0$.

$\boxed{6}$ 5. A spotlight on the ground shines on a wall 1000 cm away. A person walks from the spotlight toward the wall at a speed of 120 cm/s. At the moment when the person is 400 cm from the wall, the length of their shadow **on the wall** is decreasing at 46 cm/s. How tall is the person?



When $1000 - x = 400$, ie when $x = 600$ $\frac{dy}{dt} = -46$ cm/s.

Similar triangles: $\frac{h}{x} = \frac{y}{1000}$ (h a constant).

Thus, implicit differentiation gives:

$$-\frac{h}{x^2} \cdot \frac{dx}{dt} = \frac{1}{1000} \frac{dy}{dt}.$$

Now when $x = 600$cm, $\frac{dx}{dt} = 120$ cm/s and $\frac{dy}{dt} = -46$ cm/s.

$$\Rightarrow \frac{-h}{360\,000} \cdot 120 = \frac{1}{1000} \cdot -46$$

$$\Rightarrow -h \cdot \frac{1}{3} = -46$$

$$\Rightarrow h = 138 \text{ cm/s}.$$

$\boxed{8}$ 6. Find the absolute maximum and minimum values (if any) of the function

$$f(x) = x(21 - x^2)^{2/3}, \quad 0 \le x \le 6.$$

Derivative:

$$f'(x) = (21-x^2)^{2/3} + x(21-x^2)^{-1/3} \cdot \frac{2}{3} \cdot (-2x)$$

$$= (21-x^2)^{2/3} - \frac{\frac{4}{3}x^2}{(21-x^2)^{1/3}}$$

$$= \frac{1}{(21-x^2)^{1/3}}\left(21 - x^2 - 4/3\,x^2\right)$$

$$= \frac{1}{(21-x^2)^{1/3}}\left(21 - 7/3\,x^2\right).$$

So $f'(x) = 0 \Rightarrow 7/3\,x^2 = 21$, $f'(x)$ undef $\Rightarrow x = \sqrt{7}$

$\Rightarrow 7x^2 = 63$

$\Rightarrow x^2 = 9 \Rightarrow \boxed{x = \pm 3.}$,

So $x = 3, \sqrt{7}$ (being the only point in the domain $0 \le x \le 6$ may be an absolute max/min. We must also consider ~~???~~ the endpoints, $x = 0$, $x = 6$. Note $f(x)$ is defined everywhere in $[0,6]$, since $\sqrt[3]{}$ is defined everywhere. Testing:

$f(0) = 0$

$f(3) = 3(21-9)^{2/3} = 3 \cdot \sqrt[3]{12^2} = 3 \cdot \sqrt[3]{2^4 \cdot 3^2}$

$f(6) = 6 \cdot (21-36)^{2/3} = 6 \cdot \sqrt[3]{(-15)^2}$ ← largest

abs mins?

$= 6 \cdot \sqrt[3]{225}$ absolute max

$f(\sqrt{7}) = 0$ So the absolute mins are at $0, \sqrt{7}$, with $f(0) = f(\sqrt{7}) = 0$

7. Let $f(x) = \frac{4x}{x^2+1}$. Then $f'(x) = \frac{4(1-x^2)}{(x^2+1)^2}$, and $f''(x) = \frac{8x(x^2-3)}{(x^2+1)^3}$.

$\boxed{2}$ (a) Show that f is an odd function.

$f(-x) = \frac{4(-x)}{(-x)^2+1} = \frac{-4x}{x^2+1} = -f(x)$, so $f(x)$ is odd.

$\boxed{4}$ (b) Determine the intervals of increase/decrease for f.

$f'(x)$ changes sign (potentially) when $f'(x)=0$ or $f'(x)$ is undefined.

$f'(x)=0 \iff x^2-1=0$
$\iff x=\pm 1.$

$f'(x)$ is defined everywhere.

function	$(-\infty,-1)$	$(-1,1)$	$(1,\infty)$
$x-1$	$-$	$-$	$+$
$x+1$	$-$	$+$	$+$
$\frac{4}{(x^2+1)^2}$	$+$	$+$	$+$
$f'(x)$	$+$	$-$	$+$

$\boxed{4}$ (c) Identify all critical and singular points (if any) of f and classify them as local maxima/minima or neither.

$f(x)$ has critical points at ± 1, and no singular points since $f'(x)$ is defined everywhere. From the table above (by the first derivative test) $x=-1$ is a local max, and $x=1$ is a local min.

$\boxed{4}$ (d) Determine (if any) the interval(s) on which f is concave up and the interval(s) on which f is concave down.

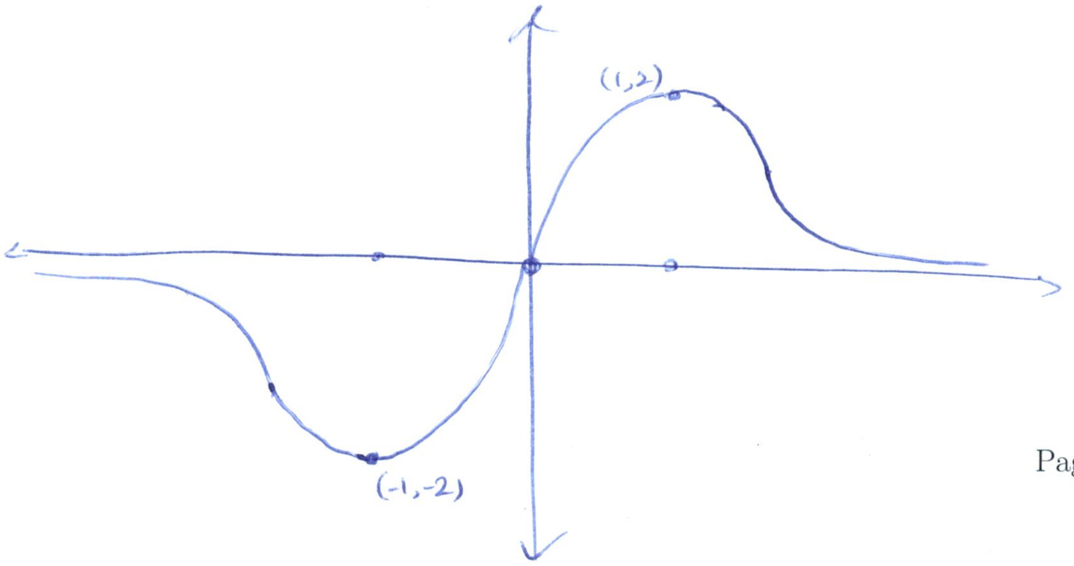
$f''(x)$ changes sign $\iff f''(x)=0$ or $f''(x)$ is undefined.

$f''(x)=0 \iff 8x(x^2-3)=0$
$\iff x=0$ or $x=\pm\sqrt{3}$

$f''(x)$ is everywhere defined.

function	$(-\infty,-\sqrt{3})$	$(-\sqrt{3},0)$	$(0,\sqrt{3})$	$(\sqrt{3},\infty)$
$8x$	$-$	$-$	$+$	$+$
$x-\sqrt{3}$	$-$	$-$	$-$	$+$
$x+\sqrt{3}$	$-$	$+$	$+$	$+$
$f''(x)$	$-$	$+$	$-$	$+$
$f(x)$	$\frown$	$\smile$	$\frown$	$\smile$

$\boxed{4}$ (e) Provide a sketch of the graph of f that exhibits all the information you found above.



4 8. (a) Calculate the Taylor polynomial $P_3(x)$ of order 3 about $x = 0$ for $f(x) = \ln(1+x)$.

Formula: $P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \cdots + \frac{f^{(n)}(a)(x-a)^n}{n!}$

So we need $f'(x) = \frac{1}{x+1}$, $f''(x) = \frac{-1}{(x+1)^2}$, $f'''(x) = \frac{2}{(x+1)^3}$

Then with $a=0$,

$$P_3(x) = \underset{\overset{\parallel}{0}}{\ln(1)} + \frac{1}{0+1}x + \frac{(-1)}{2!(0+1)^2}(x^2) + \frac{2}{3!(0+1)^3}x^3$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3}$$

1 (b) Use P_3 to write down an approximation for $\ln(2)$.

$$\ln(2) \simeq P_3(1) = 1 - \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$$

4 (c) Use Taylor's formula (i.e. Lagrange formula for the error) to estimate the error in your approximation above.

$E_3(x) = \frac{f^{(n+1)}(s)}{(n+1)!}(x-a)^{n+1}$, where s is in between x and a.

Here, $a=0$, $x=1$, and $f^{(4)}(x) = \frac{-6}{(x+1)^4}$. This has a max on $[0,1]$ at $x=0$. So

$$|E_3(1)| \le \left|\frac{f^{(4)}(0)}{4!}(1-0)^4\right| = \left|\frac{-6}{24}\right| = \frac{1}{4}.$$

5 (bonus) 9. (Bonus!) Suppose f is a differentiable function such that $f'(8) = 2$. Find the value of the limit

$$\lim_{x\to 8} \frac{f(x) - f(8)}{x^{1/3} - 2},$$

WITHOUT using L'Hopital's Rule. (If you do not know L'Hopital's Rule, do not be concerned. Solution attempts that use L'Hopital's Rule will receive no credit.)

$$\lim_{x\to 8} \frac{f(x)-f(8)}{x^{1/3}-2} = \lim_{x\to 8} \frac{f(x)-f(8)}{x-8} \cdot \frac{x-8}{x^{1/3}-2}$$

$$= \lim_{x\to 8} \frac{f(x)-f(8)}{x-8} \cdot \frac{\cancel{(x^{1/3}-2)}(x^{2/3}+2x^{1/3}+4)}{\cancel{x^{1/3}-2}}$$

$$= \lim_{x\to 8} \frac{f(x)-f(8)}{x-8} \cdot \lim_{x\to\infty} x^{2/3} + 2x^{1/3} + 4$$

$$= 2 \cdot \left[8^{2/3} + 2\,8^{1/3} + 4\right]$$

$$= 2 \cdot [4 + 4 + 4] = 24.$$