MATH 1230

§4.9 Linear approximations, start of §4.10

One way of describing the tangent line to $y=f(x)$ at $x=a$ is:

> The tangent line is the line which most (or best) approximates $f(x)$ at $x=a$.

What we mean by "best approximates" is that if the equation of the line tangent to $f(x)$ at $x=a$ is $L(x)$, then $L(x)$ is the only equation of a line with:

(i) $L(a)=f(a)$, and

(ii) $L'(a)=f'(a)$ (same first derivative)

For this reason, we say that

$$L(x)=f(a)+f'(a)(x-a) \leftarrow \text{eqn of tangent line}$$

is a linearization of $f(x)$ at $x=a$ or that it is a linear approximation of $f(x)$ at $x=a$, and we write

$$f(x) \simeq L(x).$$

These techniques/tricks can be useful in providing fast solutions to problems that are "roughly correct".

Example: Use linear approximation about $x = 25$ to estimate the value of $\sqrt{26}$.

Solution: The function $f(x)$ that we need to approximate is $f(x) = \sqrt{x}$. So we calculate the equation of the tangent line at $x = 25$:

$$f'(x) = \frac{1}{2\sqrt{x}}, \text{ therefore}$$

$$L(x) = f(25) + f'(25)(x-25)$$

$$= \sqrt{25} + \frac{1}{2\sqrt{25}}(x-25) = 5 + \frac{1}{10}(x-25).$$

So $f(26) = \sqrt{26}$ is close to

$$L(26) = 5 + \frac{1}{10}(26-25) = \frac{51}{10} = 5.1.$$

A check with a calculator gives $\sqrt{26} \simeq 5.0990\ldots$ so our approximation is fairly close.

Of course, any time we approximate something we would like to know how inaccurate our approximation is, ie we want to know roughly how big

$$\text{Error} = \text{true value} - \text{approximate value}$$

can be. In the case of linear approximation, our error is

$$E(x) = f(x) - L(x) = f(x) - (f(a) + f'(a)(x-a))$$

Theorem: If $f''(t)$ exists in an interval containing the points a and x, then there's a number s between a and x so that

$$E(x) = \frac{f''(s)}{2}(x-a)^2.$$

I.e, if we know the second derivative we can say how big our error might be.

Proof: Apply the "Generalized Mean Value Theorem" on the interval $[a,x]$ (or $[x,a]$) to the functions $E(t)$ and $(t-a)^2$. (Omit details...).

Example: Estimate the size of the error in our approximation

$$\sqrt{26} \simeq 5.1$$

without resorting to a computer.

Solution: Since the function we approximated is $f(t) = \sqrt{t}$, we need to know

$$f'(t) = \frac{1}{2}t^{-1/2}, \quad f''(t) = -\frac{1}{4}t^{-3/2}$$

in order to estimate the error.

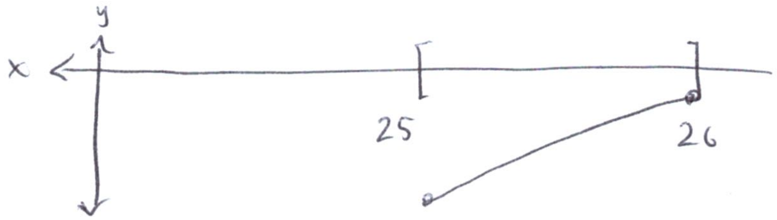
Our interval $[a,x]$ is $[25,26]$ and so

$$E(26) = \frac{f''(s)}{2}(26-25)^2 = \frac{f''(s)}{2}$$ where s is

some number in $[25,26]$. Note that our second derivative $f''(t) = -\frac{1}{4} \cdot \frac{1}{\sqrt{t^3}}$ is negative on $[25,26]$,

and increasing since
$$f'''(t) = \frac{3}{8} \cdot \frac{1}{\sqrt{t^5}} > 0 .$$

Therefore on $[25,26]$ $f''(t)$ is roughly



so $f''(t)$ is largest in absolute value at 25.

So $$|E(26)| < \left| \frac{f''(25)}{2}(26-25)^2 \right| = \frac{1}{2} \cdot \frac{1}{4} \cdot \frac{1}{\sqrt{25^3}}$$

$$= \frac{1}{8} \cdot \frac{1}{125} = \frac{1}{1000}.$$

So our estimate of $\sqrt{26} = 5.1$ has an error of at most 0.001, meanin $\sqrt{26}$ is in $[5.099, 5.101]$.

This is only a warm-up for more general calculations. Instead of asking for a best linear approximation to $f(x)$, we could ask for a best quadratic approximation to $f(x)$ at $x=a$.

This should be a $\underline{\text{quadratic}}$ function with:

$$\left.\begin{array}{ll}\text{(i)} & Q(a)=f(a)\\ \text{(ii)} & Q'(a)=f'(a)\\ \text{(iii)} & Q''(a)=f''(a)\end{array}\right\} \text{ie, first two derivatives agree}$$

What is the formula for $Q(x)$?

Well, if we're approximating $f(x)$ at $x=a$ and $Q(x)=a_0+a_1x+a_2x^2$, then

$$Q'(x)=a_1+2a_2x$$

$$Q''(x)=2a_2.$$

Therefore

$$Q''(a)=f''(a) \Rightarrow 2a_2=f''(a)$$

$$\Rightarrow a_2=\frac{f''(a)}{2}$$

$$Q'(a)=f'(a) \Rightarrow a_1+2a_2\cdot a=f'(a)$$

$$\Rightarrow a_1=f'(a)-\frac{2f''(a)}{2}\cdot a$$

$$Q(a)=f(a) \Rightarrow a_0+a_1\cdot a+a_2\cdot a^2$$

$$\Rightarrow a_0=f(a)-\underbrace{(\text{quite a mess})}=$$

$$=f(a)-\left(f'(a)-\frac{\cancel{2}f''(a)}{\cancel{2}}\cdot a\right)\cdot a+\frac{f''(a)}{2}a^2$$

Overall,

$$Q(x) = f(a) - f'(a) + \left(f'(a) - \frac{2f''(a)}{2} a\right) x + \frac{f''(a)}{2} x^2$$

$$\boxed{= f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2.}$$

Provided $f(x)$ has n derivatives at $x=a$, we can continue this pattern and find a best-approximating degree n polynomial $P_n(x)$, which satisfies:

(i) $P_n(a) = f(a)$,

(ii) $P_n^{(k)}(a) = f^{(k)}(a)$ for $k = 1, \dots, n$.

The formula for $P_n(x)$ turns out to be

$$P_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2$$

$$+ \frac{f'''(a)}{3!}(x-a)^3 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

This is called the <u>n^{th} order Taylor Polynomial of $f(x)$ at $x=a$.</u>

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§4.10 Taylor polynomials.

Last day we ended with the formula for Taylor polynomials.

Example: If $f(x)=e^x$, find $P_4(x)$ at $x=0$.

To find $P_4(x)$, we need the first four derivatives. Since it's $f(x)=e^x$, we have

$$f'(x)=e^x,\quad f''(x)=e^x,\quad f'''(x)=e^x,\quad f^{(4)}(x)=e^x, \text{ etc.}$$

Therefore

$$P_4(x) = f(0)+f'(0)(x-0)+\frac{f''(0)}{2}(x-0)^2+\frac{f'''(0)}{3!}(x-0)^3 + \frac{f^{(4)}(0)}{4!}(x-0)^4$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!}$$

Example: If $f(x) = \sin(x)$, find $P_5(x)$ at $x=0$.

Solution: We need $f^{(k)}(x)$ for $k=1,\dots,5$.

$$f'(x)=\cos(x),\quad f''(x)=-\sin(x),\quad f'''(x)=-\cos(x),\quad f^{(4)}(x)=\sin(x)$$

$$f^{(5)}(x)=\cos(x)$$

So,

$$P_5(x) = 0 + x + 0 + \frac{-1}{3!}x^3 + 0 + \frac{1}{5!}x^5$$

$$= x - \frac{1}{3!}x^3 + \frac{x^5}{5!}$$

Example: Calculate $P_2(x)$ about $x=1$ for the function $f(x) = 2x^2+1$.

Remark: This should give a curious answer. $P_2(x)$ is supposed to be the "best quadratic approximation" to $f(x) = 2x^2+1$. But $f(x)$ itself is a quadratic, so shouldn't the "best quadratic approximation" to $f(x)$ be $f(x)$ itself?

Solution: We calculate derivatives:

$$f'(x) = 4x$$

$$f''(x) = 4.$$

So then

$$P_2(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2}(x-1)^2$$

$$= 2(1)^2+1 + 4(1)(x-1) + \frac{4}{2}(x-1)^2$$

$$\boxed{= 3 + 4(x-1) + \frac{4}{2}(x-1)^2}$$

However, notice that

$$P_2(x) = 3 + 4x - 4 + \frac{4}{2}(x^2-2x+1)$$

$$= 3 + 4x - 4 + 2x^2 - 4x + 2 = \underline{2x^2+1 = f(x)}$$

So, it works!

<u>Fact</u>: The n^{th} order Taylor polynomial of a degree n polynomial is <u>the same polynomial</u>.

<u>Example (better estimation of $\sqrt{26}$)</u>

Last day we used linear approximation (ie used $P_1(x)$) to guess that $\sqrt{26} \simeq 5.1$.

What if we use $P_2(x)$?

Well, if $f(x) = \sqrt{x}$ then at $x = 25$ we compute $P_2(x)$:

$$f'(x) = \frac{1}{2\sqrt{x}} \quad \text{and} \quad f''(x) = \frac{-1}{4x^{3/2}}.$$

Therefore

$$P_2(x) = f(25) + f'(25)(x-a) + \frac{f''(25)}{2}(x-a)^2, \text{ set } x=26 \text{ and } a=25$$

$$= 5 + \frac{1}{10}(x-25) + \frac{-1}{1000}(x-25)^2$$

$$= 5.1 - \frac{1}{1000} = 5.099.$$

<u>So</u> our refined guess would be $\sqrt{26} \simeq 5.099$.

As last time, we want to estimate the error.

For this we have a theorem:

Theorem (Taylor's Theorem): Suppose $f^{(n+1)}(t)$ exists for all t in an interval containing a and x. Then the error between $P_n(x)$ and $f(x)$ is given by

$$E_n(x) = \frac{f^{(n+1)}(s)}{(n+1)!}(x-a)^{n+1} \quad \text{for some } s \text{ in between } a \text{ and } x.$$

Example: What is the largest possible error in our previous estimate of $\sqrt{26} \simeq 5.099$?

Solution: We used $P_2(x)$ for that estimation, so we need $f^{(3)}(x) = \frac{3}{8} x^{-5/2}$ to use the above theorem.

Now we find where $f^{(3)}(x)$ has the largest absolute value on $[25, 26]$. The maximum absolute value is at $x = 25$, where

$$f'''(25) = \frac{3}{8} \cdot \frac{1}{\sqrt{(25)^5}} = \frac{3}{8} \cdot \frac{1}{3125} = \frac{12}{100\,000}.$$

So

$$|E_2(26)| = \left| \frac{f^{(3)}(s)}{3!}(26-25)^3 \right| \leq \frac{12}{100\,000} \cdot \frac{1}{6} = \frac{2}{100\,000} = \frac{1}{50\,000}.$$

So $\sqrt{26} \simeq 5.099$ is accurate to four decimal places,

i.e. $\sqrt{26} \simeq 5.0990$ <u>stuff.</u>