This is a record of my talk on April 9th, 10:30am to 11:30am, at 380 Duff Roblin W380, University of Manitoba, for the Geometry and Topology Seminar.

Title: Generalized torsions in Amalgams.

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Abstract: A generalized torsion element in a group is a nontrivial element such that a product of its conjugate is trivial. An amalgam $G := A{*_C}B$ of two groups A and B, amalgamated by their common subgroup C is a group defined as following. Let A, B be the fundamental groups of two path-connected topological spaces X and Y respectively. Assume two connected open subspaces of X and Y respectively are homeomorphic with fundamental group C. Then the amalgam $G = A{*_C}B$ is defined to be the fundamental group of the disjoint union of X and Y with Z_1 and Z_2 identified.

For an amalgam $G = A *_C B$ of two subgroups amalgamated by a common subgroup C, we give a necessary and sufficient condition for the factor groups A and B to be free of generalized torsion elements of G. Combining with results of Heuer-Chen and Ito-Montegi-Teragaito, we have a sufficient condition for G to be free of generalized torsion elements. We then consider several applications of this result, including giving a Bludov-Glass type result about amalgams with matching bi-orderings and providing many groups which are free of generalized torsion element but not bi-orderable.

This is a joint work with Adam Clay.

1. Introduction, LO, BO, TF and GTF

Let's first define left-orderings and bi-orderings of group.

Definition 1.1. A subset P of a group is call a left-ordering (resp. bi-ordering) of G if $G\backslash\{1\} = P \sqcup P^{-1}$ and P is a sub-semigroup (resp. normal subsemigroup) of G.

When such orderings exists, we say the groups are left-orderable and bi-orderable, respectively. We write LO (resp. BO) for left-ordering and left-orderable (resp. bi-ordering and bi-orderable).
(The square cup refers to a disjoint union and $P^{-1} := \{a^{-1} : a \in P\}$. A subset of G is a subsemigroup if it's closed under multiplication. A subsemigroup is normal if it's closed under conjugation: $x \to g^{-1}xg$.)

Remark 1.2. People also use the equivalent language of total orders to define LO and BO: A LO (resp. BO) of a group G is a total order $<$ of the set G which is invariant under multiplication from LHS (resp. from both sides).

The correspondence is as this: Given P as in the defineion, define a total order $<$ by $a < b$ if and only if $a^{-1}b \in P$. Given such an invariant order $<$, define P by $P = \{a \in G : a > 1\}$.

Definition 1.3. A torsion (resp. generalized torsion, GT for short) in a group G is a nontrivial element $g \in G$ such that $g^n = 1$ for some $n > 1$ (resp. $h_1^{-1}gh_1 \cdots h_n^{-1}gh_n = 1$ for some $h_1, \ldots, h_n \in G$ and $n > 1$.)

We say a group is TF -meaning torsion free) (resp. GTF -meaning generalized torsion free))if it doesn't have torsion (resp. generalized torsion).

Example 1.4. (Don't know if people thought about it, but I proved the following results.)

1. In the Lie group $\mathrm{SL_n}(\mathbb{C})$, all nontrivial elements are GT.
2. So is true for $SL_2(\mathbb{R})$.
3. In the modular group $PSL_2(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z})/\{\pm 1\}$, element $T_k = \left[\begin{smallmatrix} 1 & k \\ 0 & 1 \end{smallmatrix}\right]$ is a GT if and only if $|k| \leq 5$.

1.1. Motivation.

: It is easy to see that a BO group is LO, we say BO implies LO. Similarly BO implies GTF(generalized torsion free), LO implies TF (torsion free), GTF implies TF. Also BO implies TF, consequently. All other implications is known to be false, for example, TF doesn't implies LO and there is known examples of groups which are TF but not LO; there is only one exception: it is an open problem in the well-known Kourovka Notebook: whether GTF implies LO. We believe that the answer is no, and we want to find such an example -a group which is GTF but not LO- as an amalgam of two groups. Thus we move on to the next part.

2. Amalgams of two groups

We gave a description of amalgams using Seifert-Van kampen's theorem in the abstract. We now give an algebraic definition of amalgam of two groups.

Definition 2.1. Given two group A, B written in generators and relators and relations: $A = \langle a_i; s_{i'} \rangle$ and $B = \langle b_j; r_{j'} \rangle$ (where $i \in I, i' \in I', j \in J, j' \in J$, we omit these in the expressions to ease notations.) Let $\phi : C \mapsto D$ be an isomorphism from a subgroup C of A to a subgroup D of B. Then the amalgam $G := A *_\phi B$ is defined by

$$G := A *_\phi B = \langle a_i, b_j; r_{i'}, s_{j'}, c = \phi(c), c \in C \rangle.$$

Thus the amalgam is just put the generators of A and that of B together as the generators of G, with the relations of A and that of B plus the relations $c = \phi(c)$ for $c \in C$.

Remark 2.2. It is known that $i_A : a_i \mapsto a_i$, $i_B : b_j \mapsto b_j$ give group embeddings from A, B respectively to G. Thus we view A, B as subgroups of G, also $C = D$ in G. Moreover, we also write G as $A *_C B$.

Example 2.3. Identifying the boundary of two Mobius bands, we obtain a Klein bottle. Using Seifert-Van Kampen's theorem, we have

$$\pi(Kl) = \langle a; \rangle *_{\phi} \langle b; \rangle = \langle a, b; a^2 = b^2 \rangle,$$

where $\phi : a^2 \to b^2$ gives a isomorphism from the subgroup $\langle a^2 \rangle$ of $\langle a \rangle$ to the subgroup $\langle b^2 \rangle$ of $\langle b \rangle$.

We now state Bludov-Glass's theorem (2012-2013) about the left-orderability of amalgams.

Theorem 2.4 (Bludov-Glass). *Let $G = A *_C B$ be an amalgam. Then G is LO if and only if there is a family P_i, $i \in I$, of LO of A and a family Q_j, $j \in J$, of LO of B such that*

1. *For every $i \in I$, $a \in A$, there is an $i' \in I$ such that $a^{-1}P_i a = P_{i'}$; for every $j \in J$, $b \in B$, there is a $j' \in J$ such that $b^{-1}Q_j = Q_{j'}$. (This is the **normality** condition: we say the two families are normal.)*
2. *For every $i \in I$ there is a $j \in J$ such that $P_i \cap C = Q_j \cap C$; for every $j \in J$ there is an $i \in I$ such that $P_i \cap C = Q_j \cap C$. (This is the **covering** condition: we say that the two families match.)*

3. Main result

The following is our main result, the formation of which looks similar to that of Bludov-Glass:

Theorem 3.1. *Let $G = A *_C B$ be an amalgam. Then no elements in $A \cup B$ is a generalized torsion of G if and only if there is a family M_i $i \in I$ of normal subsemigroups of A and a family N_j, $j \in J$ of normal subsemigroups of B, such that*

1. *$A \backslash \{1\} = \cup_{i \in I} M_i$, $\qquad B \backslash \{1\} = \cup_{j \in J} N_j$.*
   *(This is the **covering** condition: the the two families cover exactly the nontrivial elements or A and B respectively.)*
2. *For every $i \in I$, there is a $j \in J$, such that $M_i \cap C = N_j \cap C$; for every $j \in J$, there is an $i \in I$, such that $M_i \cap C = N_j \cap C$. (This is the **matching** condition: the two faimilies **match**.)*

If moreover C is RTF in both A and B, then G is GTF.

(Definition: A subgroup C of group A is RTF (relatively torsion free) in A if $ac_1 ac_2 \cdots ac_n \neq 1$ for all $a \in A \backslash C$ and $c_1, \ldots, c_n \in C$ $n \geq 1$.)

4. An Application of the main theorem

Here we give one example as an application of our main result.

Corollary 4.1. *Let $G = A *_C B$ C be RTF in both A and B. If there are BO P, Q of A, B respectively, such that $P \cap C = Q \cap C$, then G is GTF.*

Note that under the assumption, we know G is LO, by Bludov-Glass's theorem, even without the RTF condition. However, the amalgam G might not be BO. In fact, we have infinitely many examples of this type which is GTF but not BO. The following gives such an example.

Example 4.2. The group

$$G = \langle a, b, a', b'; a^3b^3a^5b^5 = a'^3b'^3a'^5b'^5, a^7b^{-7}a^{11}b^{-11} = a'^7b'^{-7}a'^{11}b'^{-11}\rangle$$

is GTF but not BO.

To prove that it's GTF, the main difficulty is to prove the RTF condition; we used an combinatorial argument, where we choose the prime powers so the proof is easier. To prove that it's not BO, we need a theorem of Bergman: For amalgam $G = A *_C A$ with A BO, G is BO if and only if C is relatively convex in A. (A subgroup C of A is relatively convex in A if there is a LO $<$, such that $c_1 < a < c_2$ and $c_1, c_2 \in C$ implies that $a \in C$.)