NON-BI-ORDERABILITY OF KNOT GROUPS FROM DEHN'S PRESENTATION

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Abstract. We present a computational approach to determining non-bi-orderability of knot groups based on Dehn's presentation. Our computations indicate that it may be possible to use the Alexander polynomial of a knot to prove non-bi-orderability of its knot group. This is in contrast with the result of [9], where the authors showed that knowledge of the Alexander polynomial alone is insufficient to conclude that the knot group is bi-orderable.

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1. Introduction

Given a group G, a strict total ordering $<$ of its elements is called a left-ordering if $g < h$ implies $fg < fh$ for all $f, g, h \in G$. A left-ordering of a group which is also right-invariant, in the sense that $g < h$ implies $gf < hf$ for all $f, g, h \in G$, is called a bi-ordering of G.

Motivated by the conjectured connection between orderability properties of the fundamental group and Heegaard-Floer homology, a natural class of groups to investigate from an orderability perspective are the fundamental groups of 3-manifolds [2]. This note deals specifically with the fundamental groups of knot complements—which are known to be left-orderable since they have infinite abelianization [1]—and focuses on determining when these groups are not bi-orderable.

Specifically, the purpose of this note is to demonstrate a brute force approach for proving non-bi-orderability of knot groups. Our method is similar to the approach of [3] and the appendix of [8], where the authors use a computational approach to determining non-left-orderability of the fundamental groups of certain 3-manifolds.

Our results provide computational evidence in favour of the following conjecture.

Conjecture 1.1. If K is a knot in S^3 and $\pi_1(S^3 \setminus K)$ is bi-orderable, then the Alexander polynomial $\Delta_K(t)$ has at least one positive real root.

This conjecture has already been proved in several cases. For example, if K is fibred or if K is a two-bridge knot, then it is know that $\Delta_K(t)$ must have a positive real root whenever $\pi_1(S^3 \setminus K)$ is bi-orderable [7, 6]. We add to the evidence with the following theorem.

Theorem 1.2. *Suppose that K is a knot with fewer than 10 crossings, different from 9_{49}. If $\Delta_K(t)$ has no positive real roots, then $\pi_1(S^3 \setminus K)$ is not bi-orderable.*

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The paper is organized as follows. In Section 2 we review Dehn's presentation for the purpose of establishing conventions and presenting a solution to the word problem. In Section 3 we present our algorithm for proving non-bi-orderability, and in Section 4 we state our results and provide two worked examples. In the final section we give the proof of Theorem 1.2.

2. Dehn's presentation and the word problem

Recall that Dehn's presentation of a knot group is computed from an oriented diagram of a knot K as follows. The arcs of the diagram divide the plane into regions, which we label $a, b, c, \ldots$, these will serve as the generators of the group. From the i-th crossing one creates a relator r_i by reading around the crossing, and listing the generators encountered with alternating exponents. Our convention is that for each crossing, one begins to the right of the under-arc leaving the crossing and proceeds in a clockwise manner around the crossing, listing the generators encountered with alternating signs as in Figure 1. We use capital letters in place of inverses, for ease of notation. We arrive at the presentation:

$$\langle a, b, c, \ldots \mid r_1, r_2, r_3, \ldots \rangle. \tag{2.0.1}$$

From this presentation, one arrives at Dehn's presentation by setting any one generator equal to the identity.

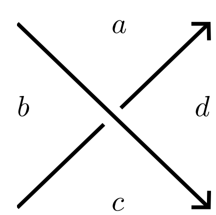

Figure 1. A crossing yielding the relation $dCbA$.

Given Dehn's presentation of a knot group, our algorithm also requires a solution to the word problem. In the case of alternating knots, we sketch the method of [5] below.

Beginning with presentation (2.0.1), colour the unbounded region of the diagram black and checkboard-colour the remaining regions. Cyclically permute the relations appearing in (2.0.1), and take inverses if necessary, so that every relator begins with a generator corresponding to a black region. If the generator corresponding to the unbounded region appears in a relator, after permuting and taking inverses the relator must begin with that generator. Then set the generator corresponding to the unbounded region equal to the identity to produce Dehn's presentation of $\pi_1(S^3 \setminus K)$.

Following [5], we then produce a complete, terminating list of rewriting rules from the relations prepared as in the previous paragraph:

(1) For every relation of length three, say UzV, find the relation of length three ending with U (such a relation always exists [5, Claim 3.3]). Say it is WyU. Then produce the following list of rules:

   (a) $v \to Uz$
   (b) $zV \to Wy$
   (c) $VU \to Z$
   (d) $ZW \to VY$

(2) For every relation of length four, say $zUyV$, we have the following list of rules:
   (a) $zU \to vY$
   (b) $yV \to uZ$
   (c) $Zv \to Uy$
   (d) $Yu \to Vz$
   (e) If there is a relation of length three ending with V, say it is XwV, then replace Rule 2a with $zU \to XwY$, and replace Rule 2c with $ZX \to UyW$.
   (f) If there is a relation of length three ending with U, say it is XwU, then replace Rule 2b with $yV \to XwZ$, and replace Rule 2d with $YX \to VzW$.

Example 2.1. Consider the knot 8_{15}, labeled as in Figure 2.

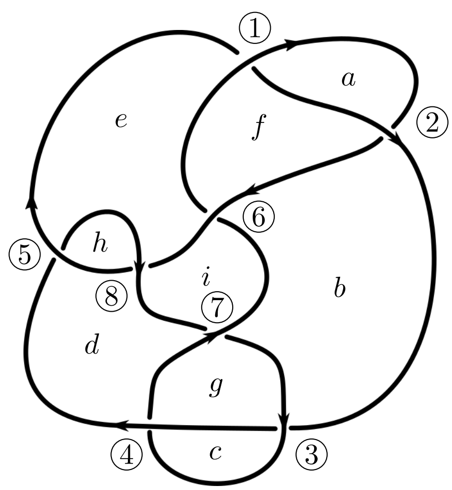

FIGURE 2. The knot 8_{15} with regions labeled and crossings numbered.

From the method above we arrive at the following relators, one from each crossing:

① AfE
② BfA
③ CgB
④ DgC
⑤ EhD
⑥ $fBiE$
⑦ $gDiB$
⑧ $hEiD$

We then produce the following rewriting system using the method above:

$e \to Af$	$a \to Bf$	$b \to Cg$	$c \to Dg$
$fE \to Bf$	$fA \to Cg$	$gB \to Dg$	$gC \to Eh$
$EA \to F$	$AB \to F$	$BC \to G$	$CD \to G$
$FB \to EF$	$FC \to AG$	$GD \to BG$	$GE \to CH$

$d \to Eh$	$fB \to AfI$	$gD \to CgI$	$hE \to EhI$
$hD \to Af$	$iE \to CgF$	$iB \to EhG$	$iD \to AfH$
$DE \to H$	$FA \to BiF$	$GC \to DiG$	$HE \to EiH$
$HA \to DF$	$IC \to EfG$	$IE \to BgH$	$IA \to DhF$

3. The algorithm

It is well known that a group G is left-orderable if and only if there exists a subset $P \subset G$, called the positive cone, satisfying

(1) $P \cdot P \subset P$
(2) $P \cap P^{-1} = \emptyset$
(3) $P \cup P^{-1} = G \setminus \{1_G\}$.

Suppose that G is finitely generated, and fix a generating set S of G. Denote the word length of an element $g \in G$ relative to S by $\ell_S(g)$. For each positive integer n, set $G_n = \{g \in G \mid \ell_S(g) \leq n\}$. If G is left-orderable with positive cone P, then for every n there exists a set $Q_n \subset G$ with

(a) $\big(Q_n \cdot Q_n\big) \cap G_n \subset Q_n$
(b) $Q_n \cap Q_n^{-1} = \emptyset$
(c) $Q_n \cup Q_n^{-1} = G_n \setminus \{1_G\}$.

For example, having fixed a positive cone P we can take $Q_n = P \cap G_n$. As a consequence, if such a Q_n does not exists for some n, then the group is not left-orderable. An algorithmic check for the existence of such a set Q_n is the basis of the computational approach to left-orderability taken in [3, 8].

This generalizes to the case of bi-orderability as follows. In addition to (1)-(3) above, the positive cone of a bi-ordering also satisfies

(4) $gPg^{-1} \subset P$ for all $g \in G$.

Let n and m be positive integers. If G is bi-orderable, then for all n and m, there exists a set $Q_{n,m} \subset G$ so that

(a) $\big(Q_{n,m} \cdot Q_{n,m}\big) \cap G_n \subset Q_{n,m}$
(b) $Q_{n,m} \cap Q_{n,m}^{-1} = \emptyset$
(c) $Q_{n,m} \cup Q_{n,m}^{-1} = G_n \setminus \{1_G\}$
(d) $g(Q_{n,m})g^{-1} \cap G_n \subset Q_{n,m}$ for all $g \in G_m$.

Remark 3.1. If $Q_{n,m}$ satisfying (a)-(d) above exists, then $Q_{n,m}^{-1}$ will also satisfy (a)-(d).

As in the case of left-orderability, if G is bi-orderable with positive cone P then $Q_{n,m} = P \cap G_n$ satisfies the properties above. Thus, if such a $Q_{n,m}$ does not exists for some n and m, then G is not bi-orderable.

Therefore the following algorithm is a test for non-bi-orderability of a group G. It takes as input integers n and m and a subset $Q \subset G_n$, it then attempts to construct a set $Q_{n,m}$ containing Q and satisfying the conditions above, and returns *false* when $Q_{n,m}$ containing Q does not exist.

function CONSTRUCTQ($Q \subset G_n$)
  while $(Q \cdot Q) \cap G_n \not\subset Q$ **do**
    $Q := (Q \cup (Q \cdot Q)) \cap G_n$
    for $g \in G_m$ **do**
      $Q := (Q \cup gQg^{-1}) \cap G_n$
    end for
  end while
  if $1_G \in Q$ **then return** false
  end if
  if $Q \cup Q^{-1} = G_n \setminus \{1_G\}$ **then return** true
  end if
  g:= a word in $G_n \setminus (Q \cup Q^{-1} \cup \{1_G\}\ )$
  return constructQ($Q \cup \{g\}$) **or** constructQ($Q \cup \{g^{-1}\}$)
end function

By Remark 3.1 and property (c), for any $g \in G_n$, we may begin by assuming $g \in Q$ (this amounts to assuming that we are constructing the positive cone of a bi-ordering in which g is positive).

In practice this function is quite slow, but it can be improved in some special cases. Focusing on the special case when G is an alternating knot group, we make two changes to improve the speed of the search.

First, every element g of the knot group G has a corresponding normal form $n(g)$ that results from iteratively applying the complete, terminating rewriting system of Section 2 to any representative word w of g (here G, implicitly is represented by Dehn's presentation). Therefore we first represent $g \in G$ by its normal form, and in place of $\ell_S(g)$ we calculate the length of every $g \in G$ by taking the word length of the normal form $n(g)$. Using normal forms and this definition of length makes it much faster to determine whether or not an element is the identity, or whether or not it is a member of G_n. Therefore, in what follows G_n consists of $g \in G$ for which the corresponding normal form $n(g)$ is a word of length less than or equal to n.

Second, we know that G is finitely generated and $G/G' \cong \mathbb{Z}$, and thus G is bi-orderable if and only if there exists a bi-ordering of G' that is invariant under conjugation by the elements of G [10]. That is, if G is bi-orderable, then G' admits a cone P' satisfying (1)-(3) above (with G replaced by G') as well as

(4') $gP'g^{-1} \subset P'$ for all $g \in G$.

So for all n and m, there must exist a set $Q'_{n,m}$ satisfying (a)-(d) above (with G_n replaced by $G'_n = G' \cap G_n$), and with (d) replaced by

(d') $g(Q'_{n,m})g^{-1} \cap G'_n \subset Q'_{n,m}$ for all $g \in G_m$.

Thus as a second improvement, we can replace all instances of G_n with G'_n, so that we have a smaller search space. Note that G'_n can be calculated from G_n by simply applying the abelianization homomorphism to every element of G_n, and keeping those elements which map to zero.

Example 3.2. When G is the knot group of 8_{15}, for example, we calculate G'_2 as follows. We begin with the Dehn presentation with relations as in Example 2.1:

$$G = \langle a, b, c, d, e, f, g, h, i \mid AfE, BfA, CgB, DgC, EhD, fBiE, gDiB, hEiD \rangle$$

whose abelianization homomorphism $\phi : G \to \mathbb{Z}$ is defined by

$$\begin{array}{ll} \phi(a) = 1 & \phi(f) = 2 \\ \phi(b) = 1 & \phi(g) = 2 \\ \phi(c) = 1 & \phi(h) = 2 \\ \phi(d) = 1 & \phi(i) = 0 \\ \phi(e) = 1. & \end{array}$$

Then we construct the list of all words which can be expressed as a product of two or fewer generators, and from the list we discard all words whose normal form is not length two. For example, under the rewriting system from Example 2.1 the word Ab becomes ACg, so it is discarded; while Fg is in normal form and so we keep it. We then apply the above abelianization homomorphism to the remaining words and discard those that do not map to zero. After these operations we find $G'_2 = \{i, I, Fg, fG, gF, Gf, Fh, fH, Hf, hF, gH, Gh, hG, Hg\}$.

4. Results

We ran our program on all alternating knots with fewer than 10 crossings, and it successfully showed that the knots 8_{15}, 9_{35}, 9_{38} and 9_{41} have non-bi-orderable knot groups. In all other cases, the program either found a subset $Q'_{n,m}$ satisfying (a)-(c) and (d') for the given n and m, or it did not terminate.

Below are the examples of 8_{15} and 9_{35}, the cases of 9_{38} and 9_{41} arein Appendix A.

Example 4.1. The knot group of 8_{15} has Dehn presentation

$$G = \langle a, b, c, d, e, f, g, h, i \mid AfE, BfA, CgB, DgC, EhD, fBiE, gDiB, hEiD \rangle.$$

Our algorithm produces the following output for $n = 3$ and $m = 2$. First, we calculate that $G'_3 = \{i, I, Fg, fG, gF, Gf, Fh, fH, Hf, hF, gH, Gh, hG, Hg, AfA, AfC, AfD, \ldots\}$, and we consider adding each of these elements in turn in an attempt to construct $Q'_{3,2}$ satisfying properties (a), (b), (c) and (d') from Section 3.

Below, each "-" represents a new instance of the function CONSTRUCTQ(Q), while indentation represents nested instances for which Q contains all of the preceding elements and is closed under properties (a) and (d'). As described in the pseudocode, *false* is returned for each instance where the identity is contained in the closure of Q under properties (a) and (d'). By Remark 3.1, we may begin by assuming that $I \in Q$. Moreover, once we have added an element to Q, taking the closure under conjugation by elements of length 3 means we do not need to test conjugate elements at subsequent steps: So, for example, once we add Fg to Q we need not carry out the test of adding gF, since $gF = g(Fg)g^{-1}$. Our program then produces the following output:

- Fg added to Q
  - Fh added to Q
    $bgfaFdHBdFhDIbfFgFeIEhgFhDiIIIdGDiFhIdIhgFhDiIIIdGHiIfFhFiDfFhIgId$ $FhDIiIGiABIbFEdDFdHBdFhDIbfFgFeIEhgFhDiIIIdGDiFhIdIhgFhDiIIIdGHiI$ $fFhFiDfFhIgIdFhDIiIGiAaFhAadgFhDiIIIdGDeGHBdFhDIbfFgFeIEhgFhDiIII$ $dGDiFhIdIhgFhDiIIIdGHiBdFhDI$ is the identity.
  - try adding Hf to Q instead
    - Hg is equivalent to $HfAaFgAa$ (thus Hg is already in Q).
      - AfA added to Q
        $aAfABIbAFAfAfEIe$ is the identity.
      - try adding aFa to Q instead
        - AfC added to Q
          $AfCcEfFgFeIC$ is the identity.
        - try adding cFa to Q instead
          - AfD added to Q
            $FfAfDFaFafFgbGDhAfDHdBgaFaGbaFgAgHfB$ is the identity.
          - try adding dFa to Q instead
            $HdFahHfBdFafaaFaAFbfHfF$ is the identity.
            Neither AfD nor dFa can be added to Q. Thus we cannot form a positive cone.
- try adding Gf to Q instead
  - Fh added to Q
    $FhcGfFIfCcfFhFfGfFCbIfFhFiIB$ is the identity.
  - try adding Hf to Q instead
    - Gh added to Q
      $cbFahCGffBgGhGbIFfHfFFBBgGhGbIbBIbgGhcGhFFBgGhGbIfFIffHgGhGA$ $BBgGhGbIbBIbfBhCGffBgGhGbIFfHfFFBBgGhGbIbBIbgGhcGhFFBgGhGbIf$ $FIffHgGhGcHFBgGhGbIfFIfhDIdCCCGffBgGhGbIFfHfFFBBgGhGbIbBIbg$ $GhcGhFFBgGhGbIfFIff$ is the identity.
    - try adding Hg to Q instead
      - AfA added to Q

- $aAfABIbAFAfAfEIe$ is the identity.
- try adding aFa to Q instead
  - AfC added to Q
  $AfCcHgCIcC$ is the identity.
  - try adding cFa to Q instead
    - AfD added to Q
    $dDAfDdHiBDAfDdbGcFagIhDHggDAfDdGgHgG$ is the identity.
    - try adding dFa to Q instead
    $HdFahHfBdFafaaFaAFbfHfF$ is the identity.
    Neither AfD nor dFa can be added to Q. Thus we cannot form a positive cone.

We conclude that $\pi_1(S^3 \setminus 8_{15})$ is not bi-orderable, in particular there is no set $Q'_{3,2}$ satisfying properties (a), (b), (c) and (d') of Section 3.

Example 4.2. In the case of the knot 9_{35}, there are ways to exploit the symmetry of the knot which allows for a proof which is nearly human-readable. Calculating a presentation of the knot group from the diagram below, we find:

$\pi_1(S^3 \backslash 9_{35}) = \langle a, b, c, d, e, f, g, h, i, j \mid BdA, AfC, ChB, eAdB, eBjA, gCfA, gAjC, iBhC, iCjB \rangle$

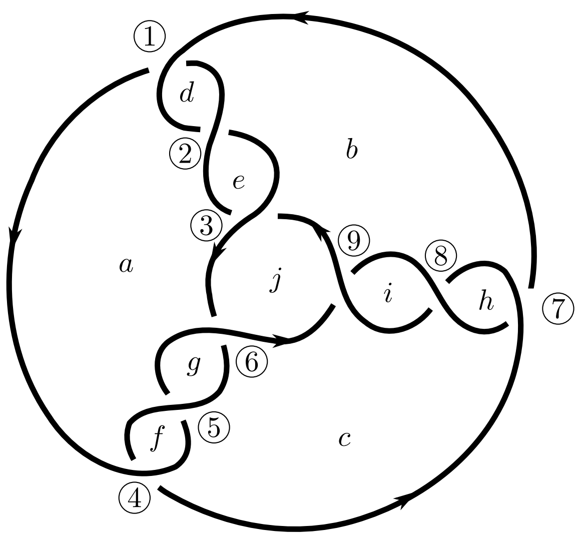

FIGURE 3. The knot 9_{35} with regions labeled and crossings numbered.

With $n = 4$ and $m = 1$ our program gives the following output. First, we calculate that $G'_4 = \{E, I, G, \ldots\}$. Begin with $E \in Q$. Then:

-G added to Q
  -I added to Q
  -$IcBEbAEaCaCIcHaGAhBGbA$ is the identity. Thus we cannot form a positive cone.
  try adding i to Q instead
    -Df added to Q
    -$cDfCEhCaGAcGHhiHbDfCEchCaGAcGHhbDDfdBiaAGaCGcAHB$

is the identity. Thus we cannot form a positive cone.
try adding Fd to Q instead
- $FfaFdABEbFcGAEaCgEGfdFdDcGCiHiIAbEBaEihIcBdFdDcGCiBEbAEaIbC$
is the identity. Thus we cannot form a positive cone.

Leaving our program to run produces roughly thirty more lines of output. However, with our program having ruled out the possibility of a positive cone Q containing either $\{E, G, I\}$ or $\{E, G, i\}$, observe that there are three automorphisms $\phi_1, \phi_2, \phi_3 : G \to G$ of order two arising from the three axes of reflective symmetry in Figure 3. Restricted to the generators e, g, i of $\pi_1(S^3 \setminus 9_{35})$, they act as:

$$\phi_1(e) = E, \phi_1(g) = I, \phi_1(i) = I$$
$$\phi_2(e) = I, \phi_2(g) = G, \phi_2(i) = E$$
$$\phi_3(e) = G, \phi_3(g) = E, \phi_3(i) = I.$$

Therefore if we suppose there exists Q containing $\{E, g, i\}$, then $\phi_2(Q)$ is a positive cone which contains $\{\phi_2(E), \phi_2(g), \phi_2(i)\} = \{E, G, i\}$, which is not possible. To rule out the final case, if Q were to contain $\{E, g, I\}$ then $(\phi_1(Q))^{-1}$ would be a positive cone which contains $\{\phi_1(E)^{-1}, \phi_1(g)^{-1}, \phi_1(I)^{-1}\} = \{E, G, i\}$, again an impossibility.

Therefore $\pi_1(S^3 \setminus 9_{35})$ is not bi-orderable.

5. Proof of Theorem 1.2

Last, we collect the necessary information to prove Theorem 1.2. First, if a knot K is either 2-bridge or fibred, and $\Delta_K(t)$ has no positive real roots, then $\pi_1(S^3 \setminus K)$ is not bi-orderable [7, 6]. Of the knots with fewer than 10 crossings, the following knots have Alexander polynomials with no positive real roots and are neither fibred nor 2-bridge, and so are not covered by either of these theorems: 8_{15}, 9_{16}, 9_{35}, 9_{38}, 9_{41}, 9_{49}. The knot group of 9_{16} admits a presentation with two generators and one tidy relator [6], and thus it is not bi-orderable [4]. Of the remaining knots, 8_{15}, 9_{35}, 9_{38} and 9_{41} have non-bi-orderable groups, with our program providing the proofs found in Section 4 and Appendix 5. The remaining knot, 9_{49}, cannot be addressed with our approach because our solution to the word problem only applies to alternating knots, and 9_{49} is not alternating.

Appendix A. The groups of 9_{38} and 9_{41} are not bi-orderable

The knot group of 9_{38} is:

$$\pi_1(S^3 \setminus 9_{38}) = \langle a, b, c, d, e, f, g, h, i, j \mid BeA, CfB, DfC, AgD, hAeB, gAhJ, hIgJ, gIfD, fIhB\rangle$$

We attempt to construct $Q := Q'_{n,m}$ with $n = 3$ and $m = 2$. The program produces the following output, assuming $E \in Q$:

-F added to Q

-G added to Q

-H added to Q

-Ai added to Q

-$bDaBfAiFbHaGAAdFbEBBdEIaAiAHieAiIfFFFiDHaHaAiAhJAijJHjAhiIDAihHHdiIFiI$ is the identity.

-try adding Ia to Q instead

-Bj added to Q

-$dFgAbBjBfIaFaGbBjBfDAHagGiGdABjhHHaDHgGIaAbEBagGIgJGjG$ is the identity.

-try adding Jb to Q instead

-$cDGbJbBjAEaJgGdCbJbBjIfIaAbEBaFFiJabbJbBjIfIaAbEBaFFiJBeJbEaIaAAgBbbJbBjIfIaAbEBaFFiJBeJbEaIaAbAEaG$ is the identity

-try adding h to Q instead

-Ai added to Q

-$aAiAaGAdFhfFDbBDAigGGdbIhfFFiIFiDAidB$ is the identity.

-try adding Ia to Q instead

-$eIaEbhBchCcDGhggGGGdCbIhfFFiIFiBfIaFaGeIaEbhBchCcDGhggGGGdCbIhfFFiIFiBgJeIaEbhBchCcDGhggGGGdCjJIajJGjaEAA$ is the identity

-try adding g to Q instead

-$CECGjHggcBFdEDbaEACcdgDECGghgDEdJjgJgcBgcBFdEDbaEACcdgDEChgDEdHbAEaIfgFicgcBFdEDbaEACcdgDEChgDEdHCeAIGjHggcBFdEDbaEACcdgDECGghgDEdJjgJgiiGdgDEgcFCcECIiEIhgDEdHhfdgDEFHacBFbaEAjAIGjHggcBFdEDbaEACcdgDECGghgDEdJjgJgiiGdgDEgcFCcECIiEIhgDEdHhfdgDEFHaJfGdgDEgcFCcECAgbEBaF$ is the identity.

-try adding f to Q instead

-G added to Q

-H added to Q

-$eAiGdCAFHbAEaBfHjfAEaJhaEfHFccCbFHbAEaBfHjfAEaJhBcfHFfCDgGfgAbEBaIaHbAEaBEeDGdCAFHbAEaBfHjfAEaJhaEfHFccCbFHbAEaBfHjfAEaJhBcfHFfCDgGfgdEbFHbAEaBfHjfAEaJhBfiffFfI$ is the identity

-try adding h to Q instead

-$BhffFbCAhgGGaAhaEcbGhgGBfAhgGGaAhaEccfGFffFfCCFiAhgGGaAhaEccfGFffFfCCFhaEfeEAfIhiI$ is the identity.

-try adding g to Q instead

-$iBdgDEbdfgFfDBhEHbBgbaEAbAEaBIhgDEdHhAFAgeEfeEEagDEdfaEHjAgeEfeEEaJgEG$ is the identity.

Thus $\pi_1(S^3 \setminus 9_{38})$ is not bi-orderable.

The knot group of 9_{41} is:

$$\pi_1(S^3 \setminus 9_{41}) = \langle a,b,c,d,e,f,g,h,i,j \mid EhA, AiB, BiC, CjD, DjE, iAhF, jFhE, jGiF, iGjC \rangle$$

We attempt to construct $Q := Q'_{n,m}$ with $n = 4$ and $m = 1$, and begin by assuming $I \in Q$. Then the program produces the following output:

-J added to Q
-Af added to Q
-$bfDDaAfAJdJdhJhAfHaIAjJHfGJgFaBAfbIAFiAfIBcDDaAfAJdJdhJhAfHaI AjJHfGJgFaBAfbIACcAEDDaAfAJdJdhJhAfHaIAjJHfGJgFjAfJeJaBAfbCA hhAfHaIAEfJFJeHabAfBcC$ is the identity.
-try adding Fa to Q instead
-AD added to Q
-AG added to Q
-Bg added to Q
-$eaEbBgBeJdABgaADDAiBgIcJCEbBgBADJbBgBADjEjBgJeiBgI$ is the identity.
-try adding Gb to Q instead
-BE added to Q
-Cf added to Q
-$hIcCfCcADCiICCfiIFaiIIcCfiHcCfiIFaiIIcCfCCCfFBEf$ is the identity.
-try adding Fc instead
-Dg added to Q
- $iaBdDgDbBEBbDgAhEDgebBEBHDgIBdDgDbBEBbDgbIDgiiIIIB$ is the identity.
-try adding Gd instead
-Ef added to Q
-$iGeEfEgIcFcCiIAEfaAJabbIEfiIFhIHfJaFaAjBbIBB$ is the identity.
-try adding Fe instead
-Ij added to Q
-$aBbIjBbIjDGbcADCdAbCgGdJBEjGcDDGbcADCdaFaAdBbhd DFcdGjGbcADCJgDHBbfJeIjDGbcADCdEaFaAjFiIjIB$ is the identity.
-try adding Ji instead
-$JiFgFfJiFdeFeEjGdJDdCgGdJBEjGcDDGbcADCdaFaAdDf GiFeIfIjJidBiJgFfJiFdeFeEjGdJDdCgGdJBEjGcDDGbcAD CdaFaAdDfGiFeIjJfJiFjIcADCbCFfJiFdeFeEjGdJDdCgGd JBEjGcDDGbcADCdaFaAdDfcCDFcdIcD$ is the identity.
-try adding eb instead
-$bhHbebBEIehFaHebBjCebcBFabbebBJCebcBFab$ is the identity.
-try adding ga instead
-$gabiIIIBcJgajJADjJC$ is the identity.
-try adding da instead
-$djFFdafFIfbiIIIBfHaFaAhJdabadaAJBDcdaCcadaACjjFFdafFIfbiIIIBfHa$

$FaAhJdabadaAJBhJadaAeJEjFiFdafFIfIfHGdjFFdafFIfbiIIIBfHaFaAhJda$ $bhdaHJBDdagJ$ is the identity.

-try adding j instead

-Af added to Q

-AD added to Q

-$aIfHBaADAaaADAaIAAbjhBAfbFjBaADAaIAbjeADdjDEJjADdjDJiJhAfHa$ $IAjiBADiIIbADIAbjBbjGjGjBaADAaIAbjeADdjDEJcADCgIcADCjiccADCCJ$ $gIjiJjB$ is the identity.

-try adding da instead

-$HjHadaAhBAfbIJIjJIhaJdaiIAfiIIjAaIAbjjHadaAhBAfbIJIjJBAfbcICJI$ $bdaBAfbBBAfbiB$ is the identity.

-try adding Fa instead

-AD added to Q

-$fhADiIIHjFaADAajAijjJjIiFaI$ is the identity.

-try adding da instead

-AG added to Q

-$caAGAadaAeIAGiFjfECjjFajjJJ$ is the identity.

-try adding ga instead

-$eCHhgaHjFaJjiIIhjcIAgaaHdjFaJjiIIDhiBFabEAgaaHdjFaJjiIIDhEjjJje$ $DdaHjFaJjiIIhaHHjFaJjiIIhEjehFaAd$ is the identity.

Thus $\pi_1(S^3 \setminus 9_{41})$ is not bi-orderable.

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