# On bi-orderability of knots

A. Akhmedov

Left-orderable groups were first introduced by Hölder.

<u>Theorem</u> (Hölder) If $G \subset Homeo_+(\mathbb{R})$, and $G$ acts freely, then $G$ is abelian.

There are of course Hölder spaces; which we know, and:

$$C^{1+\varepsilon}[0,1] = \left\{ f \in C^1[0,1] \;\middle|\; \begin{array}{c} |f'(x) - f'(y)| < K|x-y|^{\varepsilon} \\ \forall x,y \in [0,1] \end{array} \right\}.$$

Similarly $Diff^{1+\varepsilon}[0,1]$.

<u>Definition</u>: A linear order on a group $G$ is called

a) A left-order if $\forall x, y, z \in G$,

$$x < y \Rightarrow zx < zy.$$

This allows a partition of $G$, $G = G_+ \sqcup G_- \sqcup \{id\}$

b) A bi-order if $\forall x, y, z \in G$,

$$x < y \Rightarrow zx < zy \text{ and } xz < yz.$$

This also gives a partition.

c) A left-ordered group $G$ is called Archimedean if $\forall x, y \in G_+ \ \exists n$ s.t. $x^n > y$.

<u>Example</u>: $\mathrm{Homeo}_+(\mathbb{R})$ is left-orderable.

To order it, proceed as follows. Choose a dense sequence $(x_n)_{n\geq 1}$ in $\mathbb{R}$. For $f, g \in \mathrm{Homeo}_+(\mathbb{R})$ let $f < g$ if $\exists m \geq 1$ s.t.

$$f(x_i) = g(x_i) \ \forall i < m, \ \& \ f(x_m) < g(x_m).$$

Then check that

$$f < g \Rightarrow hf < hg, \text{ but not necessarily } fh < gh.$$

This group contains bi-orderable subgroups: $PL_+(I)$ and $\mathrm{Diff}_+^\omega(I)$ are BO. To see this, you can BO $\mathrm{Diff}_+^\omega$ as follows:

$$f < g \text{ if } \exists \varepsilon > 0 \text{ s.t. } f(x) < g(x) \ \forall x \in (0, \varepsilon)$$

(this works since maps in $\mathrm{Diff}_+^\omega(I)$ have finitely many fixed points).

We could generalize this to higher dimensions in several ways. One way would be to define:

<u>Def</u>: $M$ is a Hölder manifold if, whenever $\underset{\text{free}}{\Gamma} \curvearrowright M$ by homeomorphisms $\Longrightarrow \Gamma$ is abelian.

<u>E.g.</u> $\mathbb{R}$, $S^1$, and even-dimensional rational homology spheres are Hölder.

and we arrive at some open questions almost immediately:

<u>Q</u>: Is $(D^2, \partial D^2)$ Hölder?

or

<u>Q</u>: (Calegari–Rolfsen) Is $\mathrm{Homeo}_+(D^2, \partial D^2)$ left-orderable?

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Returning to orders, note we have an easy implication

LO $\Rightarrow$ no torsion, since $1 < g \Rightarrow 1 < g < g^2 < g^3 < \cdots$

A connection exists here with some of the biggest open conjectures concerning torsion-free groups. E.g.

<u>Kaplansky</u> (1940s) If $G$ is a torsion-free group and $k$ is a field, then $kG$ has no zero divisors.

This conjecture is open, but it holds when $G$ is left-orderable. The proof is to take two elements:

$$\sum_i \alpha_i g_i, \quad \sum \beta_i f_i,$$

and multiply: $\left(\sum_i \alpha_i g_i\right)\left(\sum_i \beta_i f_i\right) = \sum_i \gamma_i h_i,$

where $h_i$'s are products. But if $f_{j_e}$ is the least of all $f_j$, it gives rise to a smallest $h_i$, which cannot cancel with other terms. So the product is not zero.

Note the Kaplansky conjecture is <u>not</u> true for groups $G$ admitting torsion.

<u>Exercise</u>: If $g^n = 1$ for some $g \in G$, then check that

$$(1-g)(1+g+\dots+g^{n-1}) = 0.$$

There's a remarkable extension of Hölder's theorem:

(Akhmedov) Let $\Gamma \leq \mathrm{Diff}_+(I)$ s.t. $\exists N$ so that $\forall f \in \Gamma$, $f$ has at most $n$ fixed points. Then $\Gamma$ is meta-abelian, in fact, $\Gamma \leq \mathrm{Aff}_+ \mathbb{R}$.

$C^1$ maps

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Further relevant examples of LO and BO groups:

- Any torsion-free nilpotent group is BO.

- $\pi(\mathbb{RP}^2)$ is not LO, $\pi_1(K^2)$ is LO but not BO, and all other surface groups are BO.

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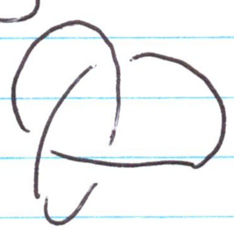
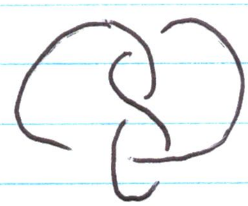
Knot groups are LO, but may be BO or not BO.

There are knot tables, with knots listed by crossing number:



3 crossings



4 crossings, etc..

The crossing number $c(L)$ is the minimal number of crossings required to draw the knot, this is how we sort knots in the tables.

Open question: $c(L_1 \# L_2) = c(L_1) + c(L_2)$ ?

We compute fundamental groups: If $K \subset S^3$, then

$\pi_1(S^3 \setminus \nu(K))$ is the knot group

E.g.

$K_{p,q}$, for $p,q$ relatively prime, is the $(p,q)$ torus knot. We have

$$\pi_1(K_{p,q}) = \langle x, y \mid x^p = y^q \rangle.$$

Then this group is not BO, for example since

powers of $x, y$ commuting $\Rightarrow$ $x, y$ commute

holds in BO groups.

Knots are called fibred if the complement $S^3 \setminus \nu(K)$ ~~bounds~~ fibres over $S^1$, or alternatively, if the Seifert surface provides a fibre for the complement.

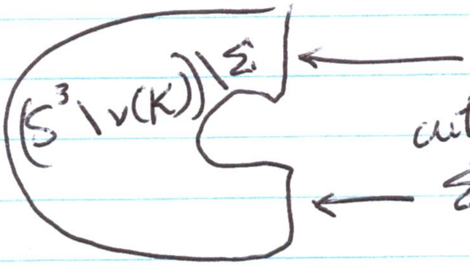
Orderability enters the picture here by investigating bi-orderability of $\pi_1(S^3 \setminus \nu(K))$.

The latest results address bi-orderability of all knots with $\leq 7$ crossings, however

$6_2$ and $7_6$ are unknown!

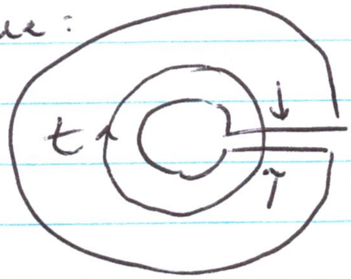
Joint work with Cody Martin: Prove that $6_2$ and $7_6$ are not bi-orderable.

Technique: Compute the knot group using the Seifert surface and "push-offs" of generators of the surface, which is essentially a Seifert-Van Kampen trick:




cutting along $\Sigma$ exposes ends, $\pi_1$ of resulting space is free.

Then re-glue:




glue to recover $S^3 \setminus \nu(K)$,

use S.V.K theorem to represent $\pi_1(S^3 \setminus \nu(K))$ as an HNN extension of a free group, with conjugating generator $t$ as shown above.

So, bi-orderability of $\pi_1(S^3 \setminus \nu(K))$ boils down to ordering the free group in a way that is invariant under the conjugation of $t$.

Then examine lower central series quotients, and the actions on them.