§15.8 March 2.

Homogeneous Linear DE's with constant coefficients

Last day we saw that: If we can somehow find n linearly independent solutions $y_1(x), y_2(x), \ldots, y_n(x)$ to an nth order homogeous linear equation, then

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x)$$

will be a general solution.

Today, we show how to find solutions $y_1(x), y_2(x), \ldots, y_n(x)$ when the linear DE has constant coefficients.

I.e.

$$a_0(x) \frac{d^n y}{dx^n} + \cdots + a_{n-1}(x) \frac{dy}{dx} + a_n(x) y = 0$$

where $a_0(x), a_1(x), \ldots, a_n(x)$ are just numbers.

Here is the method:

First, a bit of terminology:

If we solve an equation $p(x) = 0$, where $p(x)$ is a polynomial, and find that it has a factor of $(x-r)^k$ then we say r is a root of multiplicity k.

The method:

Start with a homogeneous linear DE with constant coeffs:

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_{n-1} \frac{dy}{dx} + a_n y = 0$$

or $\phi(D)y = 0$, and make the polynomial eqn $\phi(m) = 0$, ie

$$a_0 m^n + a_1 m^{n-1} + \dots + a_{n-1} m + a_n = 0.$$

Now this polynomial has a name: The auxiliary equation.

Solve the auxiliary equation and find the roots.

Then for each root:

(i) If r is a real root of multiplicity k, then

$$y(x) = (C_1 + C_2 x + \dots + C_k x^{k-1}) e^{rx}$$

is a solution to the DE.

(ii) If you find a pair of complex roots $a \pm ib$ of multiplicity k, then

$$y(x) = e^{ax}\left[(C_1 + C_2 x + \dots + C_k x^{k-1}) \cos(bx) + (D_1 + D_2 x + \dots + D_k x^{k-1}) \sin(bx) \right]$$

is a solution to the DE.

By the superposition principle + linear independence, the general solution to the DE is given by adding together all solutions found in this way.

<u>Example:</u>

Find the general solution to

$$y'' + 9y = 0.$$

<u>Solution</u>: The auxiliary equation is

$$m^2 + 9 = 0.$$

Solving, we get $m = \sqrt{-9} = \pm 3i$.

So according to part (ii), since we are dealing with a complex root $a \pm ib$ with $a = 0$, $b = 3$ and of multiplicity $k = 1$, we get a solution:

$$y(x) = e^{0 \cdot x}\left[C_1 \cancel{\sin} \cos(3x) + D_1 \sin(3x)\right]$$

$$= C_1 \cos(3x) + D_1 \sin(3x).$$

Now to get the general solution we add together all solutions we got from the various roots we found. In this case there was only $\pm 3i$, giving $y(x) = C_1 \cos(3x) + D_1 \sin(3x)$, so it's the general solution.

Example: Solve

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = 0.$$

Solution.

The auxiliary equation is $m^2 - m - 6 = 0$, which factors: $(m-3)(m+2) = 0$

$\Rightarrow r = 3, r = -2$ are roots each of multiplicity 1.

So according to our method, two linearly independent solutions are:

From $r=3$ $\quad y_1(x) = C_1 e^{3x}$

From $r=-2$ $\quad y_2(x) = C_2 e^{-2x}$.

note we need to choose different constants for each.

Now add together all solutions to get the general solution:

$$y(x) = C_1 e^{3x} + C_2 e^{-2x}.$$

Example: Find the general solution to

$$y''' - y = 0.$$

Solution: The auxiliary equation is $m^3 - 1 = 0$.

So $m=1$ is a root, and we can factor

$$m^3 - 1 = (m-1)(m^2 + m + 1) = 0.$$

so we need 2 more roots from here

Then to find the roots of m^2+m+1 we use the quadratic formula:

$$r_1, r_2 = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$= \frac{-1 \pm \sqrt{1^2-4(1)(1)}}{2(1)}$$

$$= \frac{-1 \pm \sqrt{-3}}{2}$$

$$= \frac{-1}{2} \pm i\frac{\sqrt{3}}{2}.$$

So here we have roots 1 (multiplicity one) and $\frac{-1}{2} \pm i\frac{\sqrt{3}}{2}$ (multiplicity one)

<u>From 1</u>: A solution is

$$y_1(x) = C_1 e^x.$$

<u>From $\frac{-1}{2} \pm i\frac{\sqrt{3}}{2}$</u>: A solution is

$$y_2(x) = e^{-\frac{1}{2}x}\left[C_2\cos\left(\frac{\sqrt{3}}{2}x\right) + C_3\sin\left(\frac{\sqrt{3}}{2}x\right)\right]$$

so adding these together gives the general solution:

$$y(x) = C_1e^x + e^{-\frac{1}{2}x}\left[C_2\cos\left(\frac{\sqrt{3}}{2}x\right) + C_3\sin\left(\frac{\sqrt{3}}{2}x\right)\right].$$

<u>Remark</u>: Note we used different coefficients C_1, C_2, C_3

in all of our solutions! In the final step of "adding all solutions to get the general solution", always be sure you've only used each parameter once!

Example: Solve

$$y'' + 2y' + 2y = 0, \quad y(\pi/4) = 2, \quad y'(\pi/4) = -2.$$

Solution:

The characteristic equation is

$$m^2 + 2m + 2 = 0,$$

so roots are

$$r_1, r_2 = \frac{-2 \pm \sqrt{4 - 4(1)(2)}}{2} = \frac{-2 \pm \sqrt{-4}}{2}$$

$$= -1 \pm i.$$

So the solution we get from this pair of roots is

$$y(x) = e^{-x}(c_1 \cos(x) + c_2 \sin(x)).$$

Then we need to use the initial conditions to find c_1 and c_2.

$y(\pi/4) = 2$ gives

$$2 = e^{-\pi/4}\left(c_1 \cos(\pi/4) + c_2 \sin(\pi/4)\right)$$

$$= e^{-\pi/4}\left(c_1 \frac{\sqrt{2}}{2} + c_2 \frac{\sqrt{2}}{2}\right).$$

$\Rightarrow \quad 4 = \sqrt{2}\, e^{-\pi/4}(c_1 + c_2).$ (1)

And since

$$y'(x) = c_1 e^{-x}\left(-\cos(x) - \sin(x)\right) + c_2 e^{-x}\left(\cos(x) - \sin(x)\right)$$

then $y'(\pi/4) = -2$ gives

$$-2 = c_1\left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right)e^{-\pi/4}$$

$$\Rightarrow -4 = -2c_1\sqrt{2}\, e^{-\pi/4}$$

$$\Rightarrow c_1 = \sqrt{2}\, e^{\pi/4}$$

And then (1) gives $c_2 = \sqrt{2}\, e^{\pi/4}$. So

$$y(x) = \sqrt{2}\, e^{+\pi/4}\, e^{-x}\left(\cos(x) + \sin(x)\right).$$

§ 15.9 Math 2132

Nonhomogeneous linear differential equations with constant coefficients.

Now we know how to solve

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_{n-1} \frac{dy}{dx} + a_n y = 0$$

by finding the roots of $a_0 m^n + a_1 m^{n-1} + \dots + a_{n-1} m + a_n = 0$ and using the method of last class.

The output is a general solution to the homogeneous equation above, which we could also write as

$\phi(D) y = 0.$

Next, we deal with the case $\phi(D) = F(x)$, a nonzero right hand side.

__Theorem__: Suppose we wish to find the general solution to a nonhomogeneous linear DE with constant coefficients $\phi(D) y = F(x)$.

If $y_p(x)$ is a particular solution to $\phi(D) y = F(x)$ (particular meaning it contains no constants $C_1, C_2, \dots$) and if $y_h(x)$ is a solution to $\phi(D) y = 0$

found using the techniques of last day, for example, then

$$y(x) = y_h(x) + y_p(x)$$

is the general solution to $\phi(D)y = F(x)$.

We know how to find $y_h(x)$, today we find $y_p(x)$. There are two ways to find y_p, the first is:

<u>Method 1</u>: Undetermined coefficients.

This is essentially a "guess and check" method. Here is an example to show how it works:

<u>Example</u>: Find a particular solution to

$$y'' - 4y' - 12y = 3e^{5x}.$$

What is the general solution?

<u>Solution</u>: Let's do the familiar part first: Solve for the homogeneous solution $y_h(x)$.

The DE $\quad y'' - 4y' - 12y = 0$

has complementary equation

$$m^2 - 4m - 12 = 0$$

$$\Rightarrow (m-6)(m+2) = 0$$

So the roots are $6, -2$. The general solution is

$$y_h(x) = C_1 e^{6x} + C_2 e^{-2x}.$$

Now the particular solution. We naively guess that

$$y_p(x) = Ae^{5x}$$

might work, since the right hand side is $3e^{5t}$. Well, let's find a value for 'A' that makes this work:

$$y_p'(x) = 5Ae^{5x}$$

$$y_p''(x) = 25Ae^{5x}$$

So our guess $y_p(x) = Ae^{5x}$ gives a solution if

$$25Ae^{5t} - 4\cdot 5Ae^{5x} - 12Ae^{5x} = 3e^{5x}$$

$$\Rightarrow \left(25A - 20A - 12A\right)e^{5x} = 3e^{5x}$$

$$\Rightarrow -7Ae^{5x} = 3e^{5x}$$

So $A = \frac{-3}{7}$ will work. So $y_p(x) = \frac{-3}{7}e^{5x}$

and the general solution is

$$\underline{\underline{y(x)}} = y_h(x) + y_p(x) = C_1e^{6x} + C_2e^{-2x} - \frac{3}{7}e^{5x}$$

In general, this method of guessing will always work if the function on the right hand side looks like something familiar.

There's a 'table of guesses:'

Right hand side	Guess for y_p
$\alpha e^{\beta t}$	$A e^{\beta t}$
$\alpha\cos(\beta t)$, $\alpha\sin(\beta t)$, or a sum of these	$A\cos(\beta t) + B\sin(\beta t)$
degree n polynomial	$A_n x^n + A_{n-1}x^{n-1} + \dots + A_1 x + A_0$
a sum of anything above	a sum of corresponding guesses

Example: What do you guess for $y_p(x)$ if the right hand side is:

a) $e^{2t} + 2$

Solution: This function is of the form exponential + degree 0 polynomial

So we guess $y_p(x) = Ae^{2t} + B$

b) $2\sin t - \cos 3t - t^4$

Solution: This function is of the form

$(\sin(\beta t)$ with $\beta = 1) + (\cos\beta t$ with $\beta = 3) -$ degree 4 polynomial

So we guess

$$y_p(x) = A_1\cos t + B_1 \sin t + A_2\cos 3t + B_2 \sin 3t$$
$$+ C_0 + C_1 t + C_2 t^2 + C_3 t^3 + C_4 t^4.$$

Example: Find the particular solution to

$$y'' - 4y' - 12y = t^2 + 3e^{5t}.$$

Solution: Our guess will be

$$y_p(t) = At^2 + Bt + C + De^{5t},$$

Then $y_p'(t) = 2At + B + 5De^{5t}$

$$y_p''(t) = 2A + 25De^{5t}.$$

We already worked out the value of D in the first example, and we found $D = \frac{-3}{7}$. So what remains is A, B and C which correspond to the part $At^2 + Bt + C$ of y_p. Subbing in gives:

$$2A - 4(2At + B) - 12(At^2 + Bt + C) = t^2$$

Regroup terms:

$$-12At^2 - (12B + 8A)t - 12C - 4B + 2A = t^2.$$

So equating the coefficients of powers of t:

$$-12A = 1 \Rightarrow A = \frac{-1}{12}$$

$$-12B - 8A = 0$$

$$\Rightarrow -12B = \frac{-8}{12} \Rightarrow B = \frac{1}{18}$$

$$-12C - 4B + 2A = 0$$

$$\Rightarrow -12C = \frac{4}{18} - \frac{2}{12} = \frac{1}{18} \Rightarrow C = \frac{-1}{216}$$

So $y_p(t) = \frac{-1}{12}t^2 + \frac{1}{18}t - \frac{1}{216} - \frac{3}{7}e^{5t}$.

Next we need to deal with some remaining special cases:

(i) What do you guess when the right hand side is a product? (Ans: You guess a product!)

(ii) There is a problem to be avoided in this method!

NEVER GUESS A $y_p(t)$ THAT LOOKS LIKE $y_h(t)$!

Ie. If you already solved $y_h(t)$ and found something that looks exactly like what you want to guess for y_p, then you're going to need a trick.

E.g: $y'' - 4y' - 12y = e^{-2t}$

has $y_h(t) = C_1 e^{6t} + C_2 e^{-2t}$.

We would naturally want to guess $y_p(t) = Ae^{-2t}$ but this will fail! Scale your guess by t!

(more next day)

§15.9 Undetermined coefficients continued...

Last day we saw that the general solution to a nonhomogeneous linear DE with constant coefficients can be written as

$$y(x) = y_h(x) + y_p(x)$$

and we developed a method "undetermined coeffs" for guessing the form of a particular solution $y_p(x)$, then solving for $y_p(x)$.

Today we add to our "table of guesses" with two new rules:

Rule 1: For products of functions $\alpha e^{\beta t}$, $\cos\beta t$, $\sin\beta t$, or polynomials, we guess a product of corresponding guesses. However, every product of unknown variables is replaced with a single variable.

Example: In order to solve

$$y'' + 9y = te^t$$

we solve for $y_h(t)$ first and find a complementary equation

$$m^2 + 9 = 0$$

$$\Rightarrow m^2 = -9 \Rightarrow m = \pm 3i$$

so $\quad y_h(t) = c_1 \cos(3t) + c_2 \sin(3t)$.

Then the right hand side is a product, namely

(degree one polynomial)(exponential)

so we guess a product:

$$y_p(t) = (A + Bt)\, C e^t$$

which is the same as

$$= (AC + BCt)\, e^t$$

and our rule says to replace products of unknowns with a single unknown variable. So as our "final guess" we take

$$y_p(t) = (D_1 + D_2 t)\, e^t.$$

Now solve for unknowns D_1 and D_2:

$$\left.\begin{aligned} y_p'(t) &= (D_1' + D_2' t + D_2)e^t \\ y_p''(t) &= (D_1 + D_2(t+2))e^t \end{aligned}\right\} \text{after doing some regrouping terms.}$$

Then substituting into the DE:

$$(D_1 + D_2(t+2))e^t + 9(D_1 + D_2 t)e^t = te^t$$

or $(D_1 + 2D_2 + 9D_1)e^t + (D_2 + 9D_2)\, te^t = te^t$

so $D_2 + 9D_2 = 10D_2 = 1$

$$\Rightarrow D_2 = \frac{1}{10}$$

So $10D_1 + 2D_2 = 0$

$$\Rightarrow 10D_1 = \frac{-2}{10} \Rightarrow D_1 = \frac{-2}{100}.$$

So $y_p(t) = \left(\frac{-2}{100} + \frac{1}{10}t\right)e^t$

and the general solution is $y_h(t) + y_p(t)$.

Rule 2 : Never guess a function for $y_p(t)$ that already appears as part of $y_h(t)$. If necessary, multiply your guess by powers of t until what you are guessing is not part of $y_h(t)$.

Example: Solve

$$y'' - 2y' + y = e^t.$$

Solution: First we solve for $y_h(t)$. The complementary equation is $m^2 - 2m + 1 = 0$

$$\Rightarrow (m-1)^2 = 0 \quad r=1 \text{ a root of multiplicity } 2$$

So $y_h(t) = (C_1 + C_2 t)e^t$.

Now the right hand side is e^t, so we would like to guess $y_p(t) = Ae^t$.

However, C_1e^t already appears as part of $y_h(t)$!

So in fact, if we followed through with our guess of $y_p(t) = Ae^t$, it would not work; let's see why:

$$y_p'(t) = Ae^t$$
$$y_p''(t) = Ae^t$$

Then sub in, solve for A:

$$Ae^t - 2Ae^t + Ae^t = e^t$$
$$\Rightarrow 0e^t = e^t \text{ ... problem.}$$

So, we guess $y_p(t) = Ate^t$, because the rule says to multiply our guess by powers of t. However C_2te^t is part of $y_h(t)$, so this is <u>still</u> not a good guess (subbing in would give us $0 = e^t$ again).

So multiply by another t to get

$$y_p(t) = At^2e^t$$

this does not appear as part of y_h, so it will work:

$$y_p'(t) = At^2e^t + 2Ate^t$$
$$y_p''(t) = At^2e^t + 4Ate^t + 2Ae^t$$

so sub in and get

$$\cancel{At^2e^t} + \cancel{4Ate^t} + 2Ae^t - 2(\cancel{At^2e^t} + \cancel{2Ate^t}) + \cancel{At^2e^t} = e^t$$
$$\Rightarrow 0 \cdot At^2e^t + 0 \cdot Ate^t + 2Ae^t = e^t.$$

So $2A = 1$, $A = \frac{1}{2}$.

So $y_p(t) = \frac{1}{2}t^2e^t$, and

$$y_h(t) = (C_1 + C_2 t)e^t + \frac{1}{2}t^2e^t.$$

<u>Example</u>: For the equation

$$y'' + 2y' + 2y = 3e^{-x}\sin x$$

we already solved

$$y'' + 2y' + 2y = 0$$

in an earlier example and found

$$y_h(x) = e^{-x}(C_1\cos x + C_2\sin x)$$

Here, we would be inclined to guess

$$y_p(x) = (Ae^{-x})(B\cos x + C\sin x)$$

... replacing products of coefficients ...

$$y_p(x) = e^{-x}(A\cos x + B\sin x)$$

However, our guess for y_p is then just a copy of $y_h(x)$! So we guess

$$y_p(x) = xe^{-x}(A\cos x + B\sin x).$$

Now we have to calculate y_p', y_p'' and sub into the original equation.

This is very cumbersome because of the form of y_p, but it does work out and we get:

$$B = 0 \quad A = \frac{-3}{2}, \text{ so}$$

$$y(x) = e^{-x}(c_1 \cos x + c_2 \sin x) - \frac{3}{2} x e^{-x} \cos(x)$$

is the general solution.

Next: Operator method to find y_p.

Here, the book does what is known as the "inverse operator method" to find y_p. There is another operator method called "the annihilator method". When looking at other sources, be sure not to confuse the two.

Ex: §15.8 Q1-16

Here is how the inverse operator method works:

Imagine solving

$$y'' - 2y' - 3y = e^{4x}.$$

$$\Rightarrow (D^2 - 2D - 3)y = e^{4x}.$$

We naively write:

$$y = \left(\frac{1}{D^2 - 2D - 3}\right) e^{4x}$$

Can we give meaning to the expression "$\frac{1}{D^2-2D-3}$"?

This is the goal of the inverse operator method, the result of applying $\frac{1}{D^2-2D-3}$ to e^{4x} will be y_p.