MATH 2132 §15.4

Recall that before the break, we saw how to solve 2 kinds of first order DEs:

- Separable
- linear

Then we saw how to solve some second order DE's by using substitutions that reduce them to first orders:

(i) If a second order DE contains no 'y', set $v = y'$ and $v' = y''$

(ii) If a second order DE contains no x, set

$$v = y' = \frac{dy}{dx} \quad \text{and} \quad \frac{d^2y}{dx^2} = v\frac{dv}{dy}$$

Here are some quick 'reminder examples' of how this worked.

<u>Example</u>: Solve $y'' + y' = x$.

<u>Solution</u>: Since y is missing, set $v = y'$ and $v' = y''$.

Then we get

$$v' + v = x$$

which is first order linear with $P(x) = 1$ and $Q(x) = x$

Then $\mu(x) = e^{\int P(x)dx} = e^x$, and

$$\int Q(x)\mu(x)\,dx = \int xe^x\,dx = xe^x - e^x + C_1$$

So $$v(x) = \frac{xe^x - e^x + C_1}{e^x} = x - 1 + \frac{C_1}{e^x}.$$

Now we use $v = y'$ and do:

$$y = \int v\,dx = \int x - 1 + C_1 e^{-x}\,dx = \frac{x^2}{2} - x - C_1 e^{-x} + C_2$$

Example
Solve

$$y^2 y'' - (y')^3 = 0.$$

Solution: Here there is no x, so the substitution is

$$v = y' \quad \text{and} \quad y'' = v\frac{dv}{dy}$$

We get
$$v\frac{dv}{dy}y^2 - v^3 = 0$$

$$\Rightarrow \quad \frac{dv}{dy}y^2 - v^2 = 0$$

Separate $$\frac{1}{v^2}\,dv = \frac{1}{y^2}\,dy$$

integrate $$-\frac{1}{v} = -\frac{1}{y} + C$$

or $\frac{1}{v} = \frac{1}{y} + C. \Rightarrow v = \dfrac{1}{\frac{1}{y} + C}$

Now $v = \frac{dy}{dx}$, so

$$\frac{dy}{dx} = \frac{1}{\frac{1}{y} + C}$$

Separate and integrate

$$\int \left(\frac{1}{y} + C\right) dy = \int dx$$

$$\Rightarrow \ln|y| + Cy = x + D.$$

This is an implicit form for the general solution.

There is one other well-known substitution which is commonly used to reduce a complex equation to either first order linear or separable.

An equation of the form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

is <u>not</u> linear or separable. It is a <u>Bernoulli equation</u>. There is a substitution that changes it to a first order linear equation:

Set $v = y^{1-n}$. Then

$$\frac{dv}{dx} = (1-n)y^{-n}\frac{dy}{dx}.$$

Dividing the Bernoulli equation by y^n gives

$$y^{-n}\frac{dy}{dx} + P(x)y^{1-n} = Q(x)$$

or substituting

$$\frac{1}{1-n}\frac{dv}{dx} + P(x)v = Q(x).$$

$$\Rightarrow \frac{dv}{dx} + (1-n)P(x)v = (1-n)Q(x)$$

Now the equation is linear, and we solve.

<u>Example</u>: Solve $\frac{dy}{dx} = y + y^3$

<u>Solution</u>: This is Bernoulli with $n=3$, so we set $v = y^{1-3} = y^{-2}$. Then

$$\frac{dy}{dx} - y = y^3 \qquad P(x) = -1,\ Q(x) = 1.$$

becomes, according to the formula above:

$$\frac{dv}{dx} + 2\overset{v}{\cancel{y}} = -2$$

$1-n$

Now this is linear. The integrating factor is

$$\mu(x) = e^{\int 2\,dx} = e^{2x}$$

and the solution is therefore

$$v(x) = \frac{\int e^{2x}\cdot(-2)\,dx + C}{e^{2x}}$$

$$= \frac{-e^{2x}+C}{e^{2x}} = Ce^{-2x}-1.$$

Our substitution was $v = y^{-2}$, or $y = \pm v^{-1/2}$,

so the solutions are given by

$$y = \frac{\pm 1}{\sqrt{Ce^{-2x}-1}}.$$

This completes our list of substitution tricks used to reduce an equation to first-order linear. So, we need to study a new, broader class of equations to get new results:

§ 15.6 Linear Differential equations.

An n^{th} order differential equation of the form

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^n} + \dots + a_{n-1}(x)\frac{dx}{dy} + a_n(x)y = F(x)$$

$a_0(x) \neq 0$

is called a <u>linear differential equation</u>.

It is a generalization of our first order linear equations:

$$\frac{dy}{dx} + P(x)y = Q(x)$$

and it is these equations that we will study next.

Section 15.6 Linear DE's MATH 2132.

We now begin a study of a more general class of DEs, with no restriction on the order of the equation.

A DE of the form

$$a_0(x)\frac{d^ny}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = F(x)$$

is called a <u>linear</u> DE.

Note that a linear DE <u>cannot</u> have two derivatives multiplied together, and y and its derivatives cannot appear as the argument of any functions like cos, sin, ln, etc.

We simplify notation: write "D" for the derivative with respect to x.

So $D = \dfrac{d}{dx}$

$D^2 = \dfrac{d^2}{dx^2}$

$D^3 = \dfrac{d^3}{dx^3}$, etc.

So $\dfrac{d^ny}{dx^n} = D^ny$.

With this notation, a linear DE is written as

$$a_0(x) D^n y + a_1(x) D^{n-1} y + \dots + D y \, a_{n-1}(x) + a_n(x) y = F(x).$$

or

$$\underbrace{\left(a_0(x) D^n + a_1(x) D^{n-1} + \dots + a_{n-1}(x) D + a_n(x)\right)}_{\text{this is called a differential operator, } \phi(D)} y = F(x)$$

We think of differential operators as functions that take other functions as input.

<u>Example</u>: Consider the equation

$$x^2 y'' + \cos(x) y' + y = 1.$$

What is the differential operator $\phi(D)$ corresponding to the left hand side? What is $\phi(D)$ applied to $\cos(x)$?

<u>Solution</u>: When $D = \frac{d}{dx}$, the left hand side is

$$\phi(D) = x^2 D^2 + \cos(x) D + 1$$

because when this acts on y, you get:

$$\begin{aligned}
\phi(D) y &= \left(x^2 D^2 + \cos(x) D + 1\right) y \\
&= x^2 D^2 y + \cos(x) D y + y \\
&= x^2 \frac{d^2 y}{dx^2} + \cos(x) \frac{dy}{dx} + y.
\end{aligned}$$

Applied to $\cos(x)$, this would give

$$\phi(D)\cos(x) = x^2\frac{d^2}{dx^2}(\cos(x)) + \cos(x)\frac{d}{dx}\cos(x) + \cos(x)$$

$$= x^2(-\cos(x)) + \cos(x)(-\sin(x)) + \cos(x)$$

$$= \cos(x)\left(-x^2 - \sin(x) + 1\right).$$

<u>Example</u>: Write the differential equation

$$y'' + 2y' - 3y = e^{-x}$$

in operator notation.

<u>Solution</u>: The equation becomes $\phi(D)y = e^{-x}$ where

$$\phi(D) = D^2 + 2D - 3.$$

<u>Remark</u>: When a differential equation is written in operator notation as $\phi(D)y = F(x)$, a solution to the equation is a function $y(x)$ that gives $F(x)$ as output when you plug it into the operator.

<u>Example</u>: Write the DE

$$y'' - 2y' = 5x$$

in operator notation. If the operator is $\phi(D)$, calculate $\phi(D)\left(-\frac{5}{4}(x^2+x) + e^{2x}\right)$.

Solution: The operator here is

$$\phi(D) = D^2 - 2D.$$

Then $\phi(D)\left(-\frac{5}{4}(x^2+x)+e^{2x}\right)$

$$= \frac{d^2}{dx^2}\left(-\frac{5}{4}(x^2+x)+e^{2x}\right) - 2\frac{d}{dx}\left(-\frac{5}{4}(x^2+x)+e^{2x}\right)$$

$$= \cancel{-\frac{5}{4}\cdot 2} + \cancel{4e^{2x}} - 2\left(-\frac{5}{4}\cdot 2x \cancel{-\frac{5}{4}} + \cancel{2e^{2x}}\right)$$

$$= -\frac{5}{4}\cdot(-2)\cdot 2x = 5x.$$

So $-\frac{5}{4}(x^2+x)+e^{2x}$ is a solution of the DE

Recall from linear algebra:

A function $L(x)$ is called linear if

① $L(x+y) = L(x) + L(y)$

② $L(cx) = cL(x)$

So another more formal way of describing a linear DE is:

A differential equation

$$F(x, y, y', \dots, y^{(n)}) = 0 \quad \text{is linear}$$

if the corresponding differential operator $\phi(D)$ is a linear function.

Example: By using operator notation, show that

① $y'' - 2y = 5x$ is linear, and

② $y'' + y^2 = x$ is <u>not</u> linear.

Solution: ①. This equation is <u>linear</u> because the corresponding operator $\phi(D) = D^2 - 2D$ satisfies

① $\phi(D)(y_1(x) + y_2(x))$

$$= \frac{d^2}{dx^2}\left(y_1(x) + y_2(x)\right) - 2\frac{d}{dx}\left(y_1(x) + y_2(x)\right)$$

$$= \frac{d^2}{dx^2}y_1(x) + \frac{d^2}{dx^2}y_2(x) - 2\frac{dy_1}{dx} - 2\frac{dy_2}{dx}$$

group

group

$$= \phi(D)(y_1(x)) + \phi(D)(y_2(x)),$$

② $\phi(D)(cy_1(x))$

$$= \frac{d^2}{dx^2}(cy_1(x)) + 2\frac{d}{dx}(cy_1(x))$$

$$= c\left(\frac{d^2y_1}{dx^2} - 2\frac{dy_1}{dx}\right) = c\,\phi(D)\,y_1(x).$$

② On the other hand, $y'' + y^2 = x$ is <u>not</u> linear.

The difference between this and the previous example is:

Suppose you plug $y_1(x)+y_2(x)$ into the left hand side. You get:

$$y_1''(x)+y_2''(x)+\left(y_1(x)+y_2(x)\right)^2$$
$$=y_1''(x)+y_2''(x)+y_1^2(x)+2y_1(x)y_2(x)+y_2^2(x).$$

Which is different than plugging in $y_1(x)$ and $y_2(x)$ separately, and then adding:

$$\underbrace{\left(y_1''(x)+y_1^2\right)}_{\text{plug in } y_1}+\underbrace{\left(y_2''(x)+y_2^2\right)}_{\text{plug in } y_2} \neq \text{the expression above}$$

So, from here on in our discussion of linear DE's we may choose to use operator notation when useful.

It is also useful to become accustomed to using operator notation ahead of the section on Laplace transforms, as the Laplace transform of a function is a type of linear operator as well.

§ 15.7 Homogeneous Linear DE's.

A linear DE is called <u>homogeneous</u> if the right hand side is zero when the DE is written in a standard form:

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \dots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = \underset{\uparrow \text{ homog.}}{0}$$

In operator notation, this means we're considering $\phi(D)y = 0$.

The fundamental reason for considering linear homogeneous equations separately is:

<u>Theorem 15.1</u> (Superposition principle).

If $y_1(x), y_2(x), \dots, y_n(x)$ are solutions to some homogeneous linear DE, then

$$c_1 y_1(x) + c_2 y_2(x) + \dots + c_n y_n(x)$$

is also a solution for any choice of constants $c_1, c_2, \dots, c_n$.

<u>Why is this true?</u> Remember 3 points: facts:

① A function L is linear if

$$L(x+y) = L(x) + L(y)$$

and $L(cx) = cL(x)$

② A DE is <u>linear</u> if its operator $\phi(D)$ is linear, and

③ A function $y(x)$ is a solution to $\phi(D)y = F(x)$ whenever plugging $y(x)$ into $\phi(D)$ gives $F(x)$.

So, look what happens here:

$\phi(D)(c_1y_1 + c_2y_2 + \dots + c_ny_n)$ want to check this gives 0

$= c_1\phi(D)y_1 + c_2\phi(D)y_2 + \dots + c_n\phi(D)y_n$ by fact ① and ②

$= c_1 \cdot 0 + c_2 \cdot 0 + \dots + c_n \cdot 0$ since each y_i is a solution, fact ③ gives that each is equal to 0.

$= 0$. So it's a solution to the homogeneous DE

$\phi(D)y = 0$.

<u>Terminology</u>:

If $y(x) = c_1y_1(x) + \dots + c_ny_n(x)$, then $y(x)$ is called a <u>linear combination</u> of $y_1(x), \dots, y_n(x)$.

Here is what the superposition principle does for us: If we have a linear homogeneous DE

$\phi(D)y = 0$

and somehow we come up with n solutions

$$y_1(x), y_2(x), \ldots, y_n(x),$$

then

$$c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$$

is an n-parameter family of solutions. There's a chance, then, that this n-parameter family will be a <u>general</u> <u>solution</u>.

<u>The problem</u>: Sometimes an n-parameter family doesn't contain n "genuine" degrees of freedom. For example, suppose we solve

$$y'' - y' - 12y = 0 \quad \text{by some tricks}$$

and we find solutions

$$y_1(x) = e^{4x} \quad \text{and} \quad y_2(x) = e^{4x} \cdot 10 = 10e^{4x}.$$

Then

$$\begin{aligned} y(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 e^{4x} + 10 c_2 e^{4x} \\ &= \underbrace{(c_1 + 10c_2)}_{\text{single constant}} \underbrace{e^{4x}}_{\text{function}} \end{aligned}$$

So although it appeared we had two arbitrary constants, really we only have one since

$y_1(x)$ and $y_2(x)$ are multiples of one another.

In general we need <u>linear independence</u> to get a general solution.

<u>Recall</u>: Suppose we have functions $y_1(x), y_2(x), \ldots, y_n(x)$ and

$$c_1 y_1(x) + c_2 y_2(x) + \ldots + c_n y_n(x) = 0$$

is only possible if $c_1 = c_2 = \ldots = c_n = 0$. Then $y_1(x), y_2(x), \ldots, y_n(x)$ are <u>linearly independent</u>. Otherwise the functions are <u>dependent</u>.

<u>Theorem</u>: Suppose that $y_1(x), y_2(x), \ldots, y_n(x)$ are linearly independent solutions of an nth-order linear homogeneous DE $\phi(D)y = 0$. Then the n-parameter family

$$c_1 y_1(x) + c_2 y_2(x) + \ldots + c_n y_n(x)$$

is a <u>general solution</u> to $\phi(D)y = 0$.

<u>Example</u>: Consider the differential equation

$$y'' - y = 0.$$

Just by guessing, we can come up with two solutions $y_1(x) = e^x$, $y_2(x) = e^{-x}$.

But is every solution to this DE of the form

$$y(x) = c_1 y_1 + c_2 y_2$$

for some constants c_1 & c_2 ?

Yes, says the theorem, <u>if</u> e^x and e^{-x} are linearly independent.

We check for independence:

Writing $c_1 e^x + c_2 e^{-x} = 0$ we must show this forces $c_1 = c_2 = 0$.

Well, if $c_1 e^x + c_2 e^{-x} = 0$ then plugging in any x value should still give 0:

$$x = 0 \Rightarrow c_1 e^0 + c_2 e^0 = 0$$

$$\Rightarrow c_1 + c_2 = 0$$

$$x = \ln(2) \Rightarrow c_1 e^{\ln(2)} + c_2 e^{-\ln(2)} = 0$$

plug into here:

$$\Rightarrow 2c_1 + \frac{c_2}{2} = 0.$$

$$\Rightarrow c_1 = \frac{-c_2}{4}.$$

Then

$$-\frac{c_2}{4} + c_2 = 0 \Rightarrow c_2\left(1 - \frac{1}{4}\right) = 0$$

$$\Rightarrow c_2 = 0$$

So $c_1 + c_2 = 0 \Rightarrow c_1 + 0 = 0 \Rightarrow c_1 = 0$.

So e^x and e^{-x} are linearly independent.

Therefore $y(x) = c_1 e^x + c_2 e^{-x}$ is a general solution.

So, to find general solutions to linear homogeneous DE's we will focus on methods that will give a family of linearly independent solutions $y_1(x), y_2(x), \ldots, y_n(x)$. Then the general solution will just be:

$$y(x) = c_1 y_1(x) + \ldots + c_n y_n(x).$$

Important to note:

① Our methods for solving will be engineered so as to give linearly independent solutions. So, we'll rarely have to worry about linear independence as long as we follow the procedures for solving DE's very precisely.

② If we are forced to check linear indepence, there is actually an easy method not covered in this course: The Wronskian. It is covered in question 10 of section 10.7.