§ 15.2 MATH 2132 Separable equations.

We consider only first order DE's in this section.
A differential equation is called <u>separable</u> if the y's and x's can be grouped on opposite sides of the $=$ sign.

Formally, if we can write it as

$$\frac{dy}{dx} = \frac{M(x)}{N(y)}$$

or $\underbrace{N(y)\frac{dy}{dx}}_{\text{all } y\text{'s}} = \underbrace{M(x)}_{\text{all } x\text{'s}}$ or $N(y)\,dy = M(x)\,dx$

then the equation is separable.

The technique for solving a separable equation is to separate the variables and then integrate both sides.

<u>Example:</u> Find a one-parameter family of solutions to the DE

$$\frac{dy}{dx} = \frac{3x^2}{1-y}.$$

<u>Solution:</u> Separating variables gives

$$(1-y)\,dy = 3x^2\,dx$$

$$\Rightarrow \int (1-y)\,dy = \int 3x^2\,dx$$

$$\Rightarrow y - \frac{y^2}{2} = x^3 + C.$$

Then we can write, upon multiplying the equation by -2:

$$y^2 - 2y + 2(x^3+C) = 0$$

So using the quadratic equation to solve for y, we get:

$$y = \frac{2 \pm \sqrt{4 - 4(1)(2x^3+2C)}}{2}$$

$$\Rightarrow y = 1 \pm \sqrt{4 - 8(x^3+C)}$$

$$= 1 \pm 2\sqrt{1 - 2(x^3+C)}.$$

At this point you might be asked:

Find the solution satisfying $y(0) = 3$.

Then you do:

$$3 = y(0) = 1 \pm 2\sqrt{1 - 2(0+C)}$$

$$\Rightarrow 3 = 1 \pm 2\sqrt{1-2C}$$

Now observe: $3 = 1 - 2\sqrt{1-2C}$ is not possible, so we need to use the solution

$$y = 1 + 2\sqrt{1 - 2(x^3+C)}$$

So $3 = 1 \pm 2\sqrt{1-2c}$

$$\Rightarrow c = -3/2.$$

So the particular solution satisfying $y(0)=3$ is

$y = 1 + 2\sqrt{1-2(x^3+\frac{3}{2})}$. Solution valid for $x < 2^{1/3}$.

Example: Find the general solution to

$$\frac{dy}{dt} = e^{y-t}\sec(y)(1+t^2)$$

Solution: This doesn't look like a separable equation, but it is! With some rewriting, we find:

$$\frac{dy}{dt} = e^y e^{-t} \sec(y)(1+t^2) \qquad \text{but } \sec = \frac{1}{\cos}$$

$$\Rightarrow \frac{dy}{dt} = \frac{e^{-t}(1+t^2)}{e^{-y}\cos(y)}$$

$$\Rightarrow e^{-y}\cos(y)\,dy = e^{-t}(1+t^2)\,dt$$

$$\Rightarrow \int e^{-y}\cos(y)\,dy = \int e^{-t}(1+t^2)\,dt$$

$$\Rightarrow \underbrace{\frac{e^{-y}}{2}\left(\sin(y)-\cos(y)\right)}_{\text{this is a famous trick}} = \underbrace{-e^{-t}\left(t^2+2t+3\right)}_{\text{integration by parts}} + c$$

Famous trick:

$\int e^x \cos(x)\,dx$ by parts: $u = e^x$, $du = e^x dx$, $dv = \cos(x)dx$, $v = \sin(x)$.

$$\Rightarrow \int e^x \cos(x)\,dx = e^x \sin x - \underbrace{\int \sin(x) e^x dx}_{\substack{\text{do parts again here,} \\ u = e^x,\ dv = \sin(x)dx \\ du = e^x dx \quad v = -\cos(x)}}$$

$$\Rightarrow \int e^x \cos(x)dx = e^x \sin x - \left[e^x(-\cos(x)) - \int (-\cos(x))e^x dx \right].$$

$$\Rightarrow \int e^x \cos(x)dx = e^x \sin x + e^x \cos x - \int \cos(x) e^x dx$$

$$\Rightarrow \int e^x \cos(x)dx = \frac{1}{2}\left(e^x \sin x + e^x \cos x\right).$$

So, the general solution then is

$$\frac{e^{-y}}{2}\left(\sin(y) - \cos(y)\right) = -e^{-t}(t^2 + 2t + 3) + C.$$

However it is clear that we cannot solve for y in this case. So, we must leave it in "implicit form", this is an "implicit solution".

Example: Find the solution to the initial value problem

$$y' + y^2\cos(2t) = 0, \quad y(0) = 1.$$

Solution: This is a separable equation.
If we replace y' with $\frac{dy}{dt}$, then we get

$$\frac{dy}{dt} = -y^2\cos(2t)$$

$$\Rightarrow \frac{1}{y^2}\,dy = -\cos(2t)\,dt$$

$$\Rightarrow \int \frac{1}{y^2}\,dy = -\int \cos(2t)\,dt$$

Then $\int y^{-2}\,dy = -y^{-1}$ and $\int \cos(2t)\,dt = \frac{1}{2}\sin(2t) + C$

$$\Rightarrow \frac{-1}{y} = \frac{-1}{2}\sin(2t) + C$$

or $$y = \frac{1}{\frac{1}{2}\sin(2t) + c} = \frac{2}{\sin(2t) + 2c}.$$

If we use $y(0) = 1$ we find

$$1 = y(0) = \frac{2}{\sin(0) + 2c} \Rightarrow 1 = \frac{2}{2c} \Rightarrow c = 1$$

So the solution is

$$y = \frac{2}{\sin(2t) + 2}.$$

Wednesday, Feb 11 MATH 2132 First order linear equations.

A first-order differential equation is called <u>linear</u> if it can be written as

$$\frac{dy}{dx} + P(x)y = Q(x)$$

There is a formula for solving equations of this type, so solving them is easy. The hardest part is often recognizing that the equation is first-order linear!

<u>Example</u>: The equation $\ln(x) - x^2y + xy' = 0$ is linear, because we can rewrite it as:

$$xy' - x^2y = -\ln(x)$$

$$\Rightarrow y' - xy = \frac{-\ln(x)}{x}, \quad \text{so } P(x) = -x \text{ and } Q(x) = \frac{-\ln(x)}{x}.$$

The equation $e^x \frac{dy}{dx} + 3y = x^2y$ is also linear,

because it becomes:

$$\frac{dy}{dx} + e^{-x}(3y - x^2y) = 0$$

$$\Rightarrow \frac{dy}{dx} + y(e^{-x}(3-x^2)) = 0 \quad \text{so } P(x) = e^{-x}(3-x^2), \; Q(x) = 0.$$

The solution to the equation

$$\frac{dy}{dx} + P(x)y = Q(x)$$

is

$$y = \frac{\int Q(x)e^{\int P(x)dx}\,dx + C}{e^{\int P(x)dx}}.$$

whenever P, Q are continuous.

Here is how we got this solution:

Suppose that somehow we manage to find a function $\mu(x)$ satisfying

$$\mu(x)P(x) = \mu'(x) = \frac{d\mu}{dx}.$$

Then the equation

$$\frac{dy}{dx} + P(x)y = Q(x)$$

becomes $\Rightarrow$

$$\mu(x)\frac{dy}{dx} + \mu(x)P(x)y = \mu(x)Q(x).$$

$$\Rightarrow \underbrace{\mu(x)\frac{dy}{dx} + \mu'(x)y}_{} = \mu(x)Q(x)$$

This is now the derivative of the product $\mu(x)y(x)$!

$$\Rightarrow \frac{d}{dx}\left(\mu(x)\,y(x)\right) = \mu(x)Q(x)$$

So we can just integrate both sides

integrate
$$\Rightarrow \quad \mu(x)y(x) = \int \mu(x)Q(x)\,dx$$

$$\Rightarrow \quad y(x) = \frac{\int \mu(x)Q(x)\,dx + C}{\mu(x)}.$$

So, we've solved the problem if we can find $\mu(x)$ with $\boxed{\mu'(x) = P(x)\mu(x).}$

But it is easy to check that $\mu'(x) = P(x)\mu(x)$ happens if $\mu(x) = e^{\int P(x)dx}$, by using the chain rule. So this gives the solution.

<u>Note</u>: The function $\mu(x)$ is called an <u>integrating factor</u>.

<u>Example</u>. Solve

$$\cos(x)\frac{dy}{dx} + \sin(x)y = 2\cos(x)\sin(x), \quad y\left(\tfrac{\pi}{4}\right) = 3\sqrt{2}, \quad 0 < x < \tfrac{\pi}{2}.$$

<u>Solution</u>: First we must write the equation in the form

$$\frac{dy}{dx} + P(x)y = Q(x)$$

in order to calculate the correct integrating factor.

Divide by $\cos(x)$:

$$\Rightarrow \frac{dy}{dx} + \frac{\sin(x)}{\cos(x)} y = 2\sin(x)$$

$$\Rightarrow \frac{dy}{dx} + \tan(x) y = 2\sin(x)$$

The integrating factor is $e^{\int P(x)dx} = e^{\int \tan x \, dx} = e^{\ln|\sec x|}$

$= |\sec x|$

but sec is positive if $0 < x < \pi/2$

so $\mu(x) = e^{\int P(x)dx} = \sec x.$

So now

$$y(x) = \frac{\int \mu(x) Q(x) dx + C}{\mu(x)}$$

and

$$\int \mu(x) Q(x) = \int \sec x \cdot 2\sin x = 2\int \tan(x) dx$$

$= 2 \ln|\sec x|$

$= 2\ln(\sec x)$ since $0 < x < \pi/2$.

So $$y = \frac{2\ln(\sec(x)) + C}{\sec(x)}$$

$$= \cos(x)\left(2\ln(\sec(x)) + C\right).$$

Now $y(\pi/4) = 3\sqrt{2}$, so

$$3\sqrt{2} = \cos(\pi/4)\left(2\ln(\sec(\pi/4)) + C\right)$$

$$= \frac{1}{\sqrt{2}}\left(2\ln(\sqrt{2}) + C\right)$$

$$\Rightarrow 3 = \frac{1}{2}\left(2\ln(\sqrt{2}) + C\right)$$

$$\Rightarrow C = 6 - 2\ln(\sqrt{2}) \text{ so}$$

$$y(x) = \cos(x)\left(2\ln(\sec(x)) + 6 - 2\ln(\sqrt{2})\right).$$

<u>Example.</u>

Solve

$$y' + \frac{2}{t}y = t - 1, \quad y(1) = \frac{1}{2}.$$

<u>Solution</u>: Here, $P(t) = \frac{2}{t}$ and $Q(t) = t-1$, so

and

$$\mu(t) = e^{\int P(t)dt} = e^{\int \frac{2}{t}dt} = e^{2\ln|t|} = e^{\ln|t|^2},$$

so again there is no need for the absolute values:

$$\mu(t) = e^{\ln(t^2)} = t^2.$$

Then

$$\int \mu(t)Q(t)\,dt = \int t^2(t-1)\,dt = \int t^3 - t^2\,dt$$

$$= \frac{t^4}{4} - \frac{t^3}{3}.$$

Then
$$y(t) = \frac{\int Q(t)\mu(t)\,dt + C}{\mu(t)}$$
$$= \frac{t^4/4 - t^3/3 + C}{t^2} = \frac{t^2}{4} - \frac{t}{3} + \frac{C}{t^2}.$$

Then $y(1) = \frac{1}{2}$ gives

$$\frac{1}{2} = y(1) = \frac{1}{4} - \frac{1}{3} + \frac{C}{1}$$

$$\Rightarrow C = \frac{1}{2} - \frac{1}{4} + \frac{1}{3} = \frac{6-3+4}{12} = \frac{7}{12}.$$

So $y(t) = \dfrac{t^2}{4} - \dfrac{t}{3} + \dfrac{7}{12t^2}.$

MATH 2132 §15.4

So far we've solved two kinds of first-order equations: Linear and separable.

To solve our first second-order equations, we'll consider some special kinds of second order equations that can be transformed into first-order ones.

Normally, a second order equation contains y'', y', y and x. If either:

(i) y does not appear in the equation, or

(ii) x does not appear in the equation

then we can make substitutions which reduce the second-order DE to a first-order one.

We'll do a couple of each kind. Here is (i).

<u>Example</u>: Solve $y''-2y'=5x$, $y(0)=1$ and $y'(0)=0$.

<u>Solution</u>: Since y is missing we can substitute $v=y'$, then $v'=y''$ and we get

$$v'-2v=5x,$$ a first-order linear DE.

$P(x)=-2$, $Q(x)=5x$

Then $\mu(x)=e^{-\int 2dx}=e^{-2x}$

and $\int Q(x)\mu(x)\,dx = \int 5xe^{-2x}\,dx$

$$= -\frac{5}{2}xe^{-2x} - \frac{5}{4}e^{-2x} \quad \text{(by parts).}$$

So the solution for $v(x)$ is

$$v(x) = \frac{-\frac{5}{2}xe^{-2x} - \frac{5}{4}e^{-2x} + C}{e^{-2x}}$$

$$\Rightarrow v(x) = -\frac{5}{2}x - \frac{5}{4} + Ce^{2x}$$

But $v(x) = y'(x)$, so to get $y(x)$ we integrate again:

could use $c_1 = \frac{C}{2}$ and $c_2 = D$

$$y(x) = \int v(x) = \frac{-5x^2}{4} - \frac{5x}{4} + \frac{C}{2}e^{2x} + D$$

$$= \frac{-5}{4}x^2 - \frac{5}{4}x + C_1e^{2x} + C_2.$$

Then the initial conditions (there are two now!) give:

$$1 = y(0) = 0 - 0 + C_1 + C_2 \Rightarrow 1 = C_1 + C_2$$

and $0 = y'(0) = -5/4 + 2C_1$

$$\Rightarrow 2C_1 = \frac{5}{4} \Rightarrow C_1 = \frac{5}{8} \text{ and } C_2 = 1 - \frac{5}{8} = \frac{3}{8}.$$

So $y(x) = \frac{-5}{4}x^2 - \frac{5}{4}x + \frac{5}{8}e^{2x} + \frac{3}{8}.$

Now an example of (ii), where x is missing.

In this case, we set

$$v = \frac{dy}{dx} = y' \text{ and so } \frac{dv}{dx} = \frac{dv}{dy} \cdot \overset{v}{\frac{dy}{dx}} = v\frac{dv}{dy}$$

$$\Rightarrow y'' = \frac{d^2y}{dx^2} = v\frac{dv}{dy} = vv'.$$

<u>Example</u>: Solve for the general solution to

$$y'' = 4(y')^{3/2}y.$$

<u>Solution</u>: Set $v = y'$, then $y'' = v\frac{dv}{dy}$ and

$$v\frac{dv}{dy} = 4v^{3/2}y \quad \text{first order in } v \text{ and } y$$

$$\Rightarrow \frac{1}{\sqrt{v}}\,dv = 4y\,dy \quad \text{it's separable, integrate both sides}$$

$$\Rightarrow \frac{2}{\cancel{3}}v^{1/2} = 2y^2 + C$$

$$\Rightarrow v^{1/2} = y^2 + C_1 \quad \text{where } C_1 = \frac{C}{2}.$$

However, $v = \frac{dy}{dx}$ so we get

$$\sqrt{\frac{dy}{dx}} = y^2 + C_1$$

$$\Rightarrow \frac{dy}{dx} = (y^2 + C_1)^2, \quad \text{now we need to solve this for } y(x)$$

separate again

$\Rightarrow$
$$\frac{1}{(y^2+C_1)^2}\,dy = dx$$

integrate: $$x = \int \underbrace{\frac{1}{(y^2+C_1)^2}\,dy}_{}$$

quite tricky, must use trig substitutions! You do different substitutions depending on whether $C_1>0$, $C_1<0$ (or $C_1=0$) and so get different answers in each case:

$$x = \begin{cases} \frac{1}{2}C_1^{-3/2}\left(\tan^{-1}\left(\frac{y}{C_1^{1/2}}\right) + \frac{\sqrt{C_1}\,y}{y^2+C_1}\right) + C_2 & \text{if } C_1>0 \\ -\frac{1}{3y^3} + C_2 & \text{if } C_1 = 0 \\ -\frac{1}{2(-C_1)^{3/2}} \cdot \frac{y^2}{y^2+C_1} + C_2 & \text{if } C_1 < 0. \end{cases}$$

So this horrible thing is the general solution

Example: Another example with no 'x' appearing.

Solve $y'' = (y')^2$

Solution: Set $v = y'$ and $y'' = v\,v'$. Then ($v' = \frac{dv}{dy}$)

$$v\frac{dv}{dy} = v^2. \Rightarrow \frac{1}{v}\,dv = dy$$

provided $v \neq 0$. So integrate:

$$\ln|v| = y + c.$$

take exponentials of both sides

$$|v| = e^{y+c} = e^c \cdot e^y$$ set $C_1 = e^c$, a new constant.

So $|v| = \left|\frac{dy}{dx}\right| = C_1 e^y$ or

$$\frac{dy}{dx} = \pm C_1 e^y,$$ "absorb" the $\pm$ into the arbitrary constant C_1

$$\Rightarrow \quad \frac{dy}{dx} = C_1 e^y.$$

Separate again

$$e^{-y}\,dy = C_1\,dx$$

integrate $\Rightarrow$ $-e^{-y} = C_1 x + C_2.$

Now solve for y if we can, here this means taking $\ln$ of both sides:

$$\Rightarrow \ln(+e^{-y}) = \ln(-C_1 x - C_2)$$

$$\Rightarrow \quad y = -\ln(-C_1 x - C_2)$$ again we could just write

$$y = -\ln(C_1 x + C_2)$$ if we think of "absorbing signs" into the arbitrary constants.