MATH 2132 Feb 2 §10.6 ctd.

Today, a bit more practice summing series.

Including a tip:

If you are asked to find the sum $\sum_{n=0}^{\infty} a_n x^n$, set it equal to $S(x) = \sum_{n=0}^{\infty} a_n x^n$ then use the name $S(x)$ to refer to it and 'track changes' as you manipulate the sum.

Eg. Question 4 §10.6.

Find the sum of $\sum_{n=1}^{\infty} n^2 x^{n-1}$.

Solution: Integrating will give us some cancelation. So we try that:

$$S(x) = \sum_{n=1}^{\infty} n^2 x^{n-1}$$

$$\Rightarrow \int S(x)\,dx = \sum_{n=1}^{\infty} \int n^2 x^{n-1}\,dx$$

$$= \sum_{n=1}^{\infty} n x^n + C$$

Now observe:

If the power on x was $n-1$, we could integrate and get cancellation again! We can make the power into what we want:

$$\int s(x) = x \sum_{n=1}^{\infty} n x^{n-1} + C$$

$$\Rightarrow \int\int s(x)dx = \int x \sum_{n=1}^{\infty} n x^{n-1} \, dx + \int C \, dx.$$

Now integrate by parts! Set $u = x$, $dv = \sum_{n=1}^{\infty} n x^{n-1} dx$.

The $\int u\,dv = uv - \int v\,du$, so we need to know

$$v = \sum_{n=1}^{\infty} \int n x^{n-1} dx = \sum_{n=1}^{\infty} x^n = \sum_{n=0}^{\infty} x^n - 1$$

and $du = dx$.

$$= \frac{1}{1-x} - 1.$$

So

$$\int\int s(x)\,dx = x \cdot \left(\frac{1}{1-x} - 1\right) - \int\left(\frac{1}{1-x} - 1\right)dx + Cx + D$$

$$= \frac{x}{1-x} - x - \int\left(\frac{1}{1-x} - 1\right)dx + Cx + D.$$

We've integrated twice to get here, so to get back to $s(x)$ we differentiate twice:

$$\int s(x)\,dx = \frac{d}{dx}\left(\frac{x}{1-x} - 1\right) - \frac{d}{dx}\int\left(\frac{1}{1-x} - 1\right)dx + C.$$

$$= \frac{1}{(1-x)^2} - \frac{1}{1-x} - 1 + C.$$

and again:

$$s(x) = \frac{d}{dx}\left(\frac{1}{(1-x)^2} - \frac{1}{1-x} - 1 + c\right)$$

$$= \frac{+2}{(1-x)^3} - \frac{1}{(1-x)^2}$$

$$= \frac{2}{(1-x)^3} - \frac{1-x}{(1-x)^3} = \frac{x+1}{(1-x)^3}$$

The only formula we used was $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$,

which is valid on $-1<x<1$. So our result converges on the same interval: $-1<x<1$.

<u>Example</u>: Q 14, § 10.6

Find the sum of

$$\sum_{n=1}^{\infty} \frac{(2n+3)2^n}{n!} x^{2n}$$

Solution: Same kind of trick. We have

$$S(x) = \sum_{n=1}^{\infty} \frac{(2n+3)2^n}{n!} x^{2n}$$ ← wouldn't it be nice if this were 2n+2? Then integrate and cancel...

$$= x^{-2} \sum_{n=1}^{\infty} \frac{(2n+3)2^n}{n!} x^{2n+2}$$

So then integrating:

$$\int S(x) = \int x^{-2} \sum_{n=1}^{\infty} \frac{(2n+3)2^n}{n!} x^{2n+2}\, dx. + C$$

By parts: $u(x) = x^{-2}$, $dv = \sum_{n=1}^{\infty} \frac{(2n+3)2^n}{n!} x^{2n+2} dx$

So $du = -2x^{-3}dx$, $v = \sum_{n=1}^{\infty} \frac{2^n}{n!} x^{2n+3}$

$$= \sum_{n=1}^{\infty} x^3 \frac{(2x^2)^n}{n!}$$

$$= x^3 \sum_{n=1}^{\infty} \frac{(2x^2)^n}{n!}$$

However this formula for v looks very familiar, recall $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. So

$$v = x^3\left(\sum_{n=0}^{\infty} \frac{(2x^2)^n}{n!} - 1\right) = x^3 e^{2x^2} - x^3.$$

Therefore using $\int u\,dv = uv - \int v\,du$

$$\int s(x)\,dx = uv - \int v\,du + C$$

$$= \frac{1}{x^2}\left(x^3e^{2x^2} - x^3\right) - \int \left(x^3e^{2x^2} - x^3\right)\left(-2x^{-3}dx\right) + C$$

$$= xe^{2x^2} - x - \int -2\left(e^{2x^2} - 1\right)dx + C$$

So we differentiate to get back to $s(x)$:

$$s(x) = \frac{d}{dx}\left(xe^{2x^2} - x\right) + \frac{d}{dx}\int 2\left(e^{2x^2} - 1\right)dx$$

$$= e^{2x^2}(4x^2+1) - 1 + 2e^{2x^2} - 2$$

$$= (4x^2+3)e^{2x^2} - 3$$

Now the formula we used to find v was

$\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$, which is valid of all values of x $(R=\infty)$.

So the series we have just analyzed converges for all x.

Section 10.6 #6

Find the sum of $\sum_{n=0}^{\infty} \frac{1}{n+1} x^n$

Solution: Recall that

$$\ln(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (x-1)^n, \quad (0 < x < 2) \text{ so}$$

$$\ln(1-x) = -\sum_{n=1}^{\infty} \frac{+x^n}{n}, \quad -1 < x < 1.$$

Now observe that

$$\sum_{n=0}^{\infty} \frac{1}{n+1} x^n = \frac{1}{x} \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$$

ie. replacing $n+1$ by "m" this is

$$\frac{1}{x} \sum_{m=1}^{\infty} \frac{x^m}{m}$$

$$= \frac{1}{x}\left(-\ln(1-x)\right), \quad -1 < x < 1$$

$$= \frac{-\ln(1-x)}{x} \text{ for } -1 < x < 1.$$

So sometimes you don't need integration or differentiation, despite the book saying that you do.

§10.7 Applications of Taylor series and the Remainder formula.

The Taylor polynomials (and Taylor series) of a function $f(x)$ allow us to approximate $f(x)$ by polynomials:

$$f(x) \simeq f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \dots + \frac{f^{(n)}(c)}{n!}(x-c)^n$$

$$+ \underbrace{\frac{f^{(n+1)}(z_n)}{(n+1)!}(x-c)^{n+1}}_{\text{Remainder}}$$

So basically:

Suppose you have a function $f(x)$, and some complicated computation you want to do with $f(x)$. If you can't do the computation directly using the formula for $f(x)$ but you can do it for polynomials, then using Taylor series you can get an answer as close to the actual answer as you like.

Example:

Evaluate the integral $\int_0^{1/2} \frac{\sin(x)}{x}\,dx$.

Solution: Normally, we would do

$$\int_a^b f(x)\,dx = F(b) - F(a),$$ where F is an antiderivative of f. However $\frac{\sin(x)}{x}$ does not have an antiderivative that can be expressed in terms of basic functions like cos, sin, e^x, etc.

However we can evaluate the integral as precisely as we want:

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \cdots$$ where x is in $(0, \frac{1}{2})$

and $$R_n = \frac{f^{(n+1)}(z_n)}{(n+1)!} x^n,$$ here $f(x) = \sin(x)$ so

$$R_n = \left.\frac{d^{n+1}}{dx^{n+1}}(\sin(x))\right|_{x=z_0} \cdot \frac{x^n}{(n+1)!}$$

So overall, $\frac{\sin(x)}{x}$ is replaced by

$$\frac{\sin(x)}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots + \frac{1}{x} R_n.$$

So if we want to know the value of the integral above, we can do:

$$\int_0^{1/2} \frac{\sin x}{x}\,dx = \int_0^{1/2} 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots\,dx + \int_0^{1/2} \frac{1}{x} R_n\,dx$$

So we just integrate the first part as we would any polynomial:

$$= \left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \right]_0^{1/2} + \int_0^{1/2} \frac{R_n}{x}\, dx.$$

$$= \underbrace{\frac{1}{2} - \frac{\left(\frac{1}{2}\right)^3}{3\,3!} + \frac{\left(\frac{1}{2}\right)^5}{5\,5!} - \frac{\left(\frac{1}{2}\right)^7}{7\,7!} + \cdots}_{\text{This is just some number}} + \underbrace{\int_0^{1/2} \frac{R_n}{x}\, dx.}_{\text{this is the error in our answer.}}$$

How big is the error? Well,

$$R_n = \frac{f^{(n+1)}(z_n)}{(n+1)!} x^n$$ where $f^{(n+1)}(z_n)$ is either $\pm \cos$, or $\pm \sin$

In either case, $|f^{(n+1)}(z_n)| \leq 1$. So

$$|R_n| \leq 1 \cdot \frac{x^n}{(n+1)!}$$

Therefore our error $\int_0^{1/2} \frac{R_n}{x}\, dx$ is smaller than:

$$\left| \int_0^{1/2} \frac{R_n}{x}\, dx \right| \leq \int_0^{1/2} \frac{1}{x} \cdot \frac{x^n}{(n+1)!}\, dx$$

$$= \int_0^{1/2} \frac{x^{n-1}}{(n+1)!}\, dx$$

$$= \left[\frac{x^n}{(n+1)(n+1)!} \right]_0^{1/2} = \frac{\left(\frac{1}{2}\right)^{n+1}}{(n+1)(n+1)!}$$

So let's use some actual numbers: If we use 3 nonzero terms of the Taylor expansion, we get

$$\int_0^{1/2} \frac{\sin x}{x}\,dx \cong \int_0^{1/2} \frac{1}{x}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!}\right)dx$$

$$\cong 0.49310763 9.$$

and the error in our calculation is at most

$$\int_0^{1/2} \frac{R_5}{x}\,dx \leq \frac{\left(\frac{1}{2}\right)^{5+1}}{6!\cdot 6} = 0.000003616.$$

Here is another "application".

Often computers, calculators, etc give a repeating decimal output to a calculation. If a decimal repeats it is always equal to a fraction.

Example: Write

$$3.2\overline{394}394394\ldots \quad \text{as a fraction.}$$

Solution: Let us write out ~~wat~~ what this decimal means. It is

$$3\times 1 + 2\times 0.1 + 3\times 0.01 + 9\times 0.001 + 4\times 0.0001 + 3\times 0.00001 + \ldots$$

ie. $$3 + 2\cdot\frac{1}{10} + 3\times\frac{1}{10^2} + 9\times\frac{1}{10^3} + 4\times\frac{1}{10^4} + 3\times\frac{1}{105} + \ldots$$

$$= 3 + \frac{2}{10} + 3\sum_{n=0}^{\infty}\frac{1}{10^{2+3n}} + 9\sum_{n=0}^{\infty}\frac{1}{10^{3+3n}} + 4\sum_{n=0}^{\infty}\frac{1}{10^{4+3n}}$$

$$= 3 + \frac{2}{10} + \frac{3}{100}\left(\sum_{n=0}^{\infty}\left(\frac{1}{10^3}\right)^n\right) + \frac{9}{1000}\left(\sum_{n=0}^{\infty}\left(\frac{1}{10^3}\right)^n\right)$$

$$+ \frac{4}{10000}\left(\sum_{n=0}^{\infty}\left(\frac{1}{10^3}\right)^n\right).$$

What is the sum $\sum_{n=0}^{\infty}\left(\frac{1}{10^3}\right)^n$? This is $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$, with $x = \frac{1}{10^3}$. So

$$\sum_{n=0}^{\infty}\left(\frac{1}{10^3}\right)^n = \frac{1}{1-\frac{1}{10^3}} = \frac{1}{\frac{999}{1000}} = \frac{1000}{999}.$$

So

$$3.2\overline{394} = 3 + \frac{2}{10} + \frac{3}{100}\left(\frac{1000}{999}\right) + \frac{9}{1000}\left(\frac{1000}{999}\right) + \frac{4}{10000}\left(\frac{1000}{999}\right)$$

$$= 3 + \frac{2}{10} + \frac{30}{999} + \frac{9}{999} + \frac{4}{9990}.$$

$$= \frac{16181}{4995}.$$

(This is a question from last year's exam).

<u>Evaluating limits</u>: We can also easily evaluate some limits.

<u>Example</u>: What is $\lim_{x \to 0} \frac{\sin x}{x}$?

<u>Solution</u>: Well, it's 1. But how to show this?

If we change to power series:

$$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{1}{x}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)$$

$$= \lim_{x\to 0} 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots$$

$$= 1 - 0 + 0 - 0 + 0 \cdots = 1.$$

<u>Example</u>: Evaluate

$$\lim_{x\to 0} \frac{\sin x - x}{x^3}.$$

<u>Solution</u>: Replace $\sin x$ with its power series. Then

$$\lim_{x\to 0} \frac{\sin x - x}{x^3} = \lim_{x\to 0} \frac{1}{x^3}\left(\cancel{x} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots - \cancel{x}\right)$$

$$= \lim_{x\to 0} \frac{1}{x^3}\left(-\frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)$$

$$= \lim_{x\to 0} -\frac{1}{3!} + \cancelto{0}{\frac{x^2}{5!}} - \cancelto{0}{\frac{x^4}{7!}} + \cancelto{0}{\cdots}$$

$$= \frac{-1}{3!} = \frac{-1}{6}.$$

Differential equations.

We can also use Taylor series to solve differential equations.

Since the rest of the course is about differential equations, we'll postpone this discussion until we have a lot of DE's under our belt.

MATH 2132 § 15.1 Introduction to differential equations and terminology

A differential equation (DE) is an equation that contains an unknown function, usually denoted y or $f(x)$, and its derivatives.

E.g. $\frac{dy}{dx} + \cos(x)y = \sin(x)$ unknown function $y = y(x)$

or $10f^{(4)}(x) + 7f^{(3)}(x) - f(x) = 0$, unknown function $f(x)$

or $\frac{d^5y}{dx^5} = 0$.

To solve a differential equation means to find a function whose derivatives obey the equation. There are often many different solutions to a DE.

Example: Show that $y(t) = t^{3/2}$ is a solution to the DE $2t^2y'' + ty' - 3y = 0$.

Solution: If $y(t) = t^{3/2}$, then $y'(t) = y' = \frac{3}{2}t^{1/2}$; and

~~Then if we plug in, we get:~~ $y''(t) = y'' = \frac{3}{4}t^{-1/2}$.

$\cancel{2t^2\left(\frac{3}{2}t^{1/2}\right) + t\left(\frac{3}{2}t^{1/2}\right) - 3t^{3/2}}$

$\cancel{= \frac{3}{\cancel{2}}t^{3/2} + \frac{3}{2}t^{3/2}}$

Then we plug in and get

$$2t^2\left(\frac{3}{4}t^{-1/2}\right) + t\left(\frac{3}{2}t^{1/2}\right) - 3t^{3/2}$$

$$= \frac{3}{2}t^{3/2} + \frac{3}{2}t^{3/2} - 3t^{3/2} = 0.$$

One can check that $y(t) = \frac{1}{t}$ is also a solution.

More terminology: The <u>order</u> of a DE is the order of the highest derivative appearing in the equation. So for example, the equation $\frac{d^5y}{dx^5} = 0$ is fifth order, whereas the equation from the last example is second order.

As already pointed out, a DE can have many solutions. To cut down the possible answers from "many" to "one", DE's often come with a bit of supplemental data called <u>initial conditions</u>:

<u>Example</u>: Find a solution to

$$2t^2y'' + ty' - 3y = 0$$

that satisfies $y''(0) = 0$, and $y'(0) = 0$.

Solution: We calculated already that $y(t) = t^{3/2}$ is a solution, and were told that $y(t) = \frac{1}{t}$ is also a solution. Of these two possibilities, only $y(t) = t^{3/2}$, which has $y'(t) = \frac{3}{2}t^{1/2}$ and $y''(t) = \frac{3}{4}t^{-1/2}$ satisfies $y''(0) = y'(0) = 0$. So one such solution is $y(t) = t^{3/2}$.

Since a DE can have many possible solutions, we often want to describe all the possible solutions to a DE in a concise way. This is done using n-parameter families of solutions when the DE has order n.

For example in our previous equation, both $y_1(t) = t^{3/2}$ and $y_2(t) = \frac{1}{t}$ are solutions, and in fact so is every function of the form

$$\underbrace{c_1 \, y_1(t) + c_2 \, y_2(t)}_{}, \quad c_1, c_2 \text{ constants},$$

parameters.

this is a 2-parameter family of solutions

If every solution to some differential equation is a member of a given n-parameter family, then the n-parameter family is called the general solution

Example: It turns out that every solution to

$$2t^2y'' + ty' - 3y = 0$$

can be written as

$$y(t) = c_1 t^{3/2} + c_2 \frac{1}{t}$$

for some choice of constants c_1, c_2. So this is called a general solution.

Example: Find a general solution to

$$\frac{d^2y}{dx^2} = xe^{-x}.$$

Solution: We can isolate for y in this equation by integrating twice:

$$\int \frac{d^2y}{dx^2} = \frac{dy}{dx} = \int xe^{-x}\,dx = -xe^{-x} - e^{-x} + C_1$$

Then integrate again:

$$\int \frac{dy}{dx}\,dx = y = \int -xe^{-x} - e^{-x} + C_1\,dx$$

$$= xe^{-x} + 2e^{-x} + C_1 x + C_2.$$

Observe that the constants of integration now become parameters in a 2-parameter family of solutions to this DE. In fact, every solution to this DE is of this form, so it's a general

Solution.

Last bit of terminology: Once we have a general solution, like

$$y(t) = c_1 y_1(t) + c_2 y_2(t),$$

making a particular choice of constants like:

$c_1 = 2 \quad c_2 = -5$

results in $\quad y(t) = 2y_1(t) - 5y_2(t),$

and this is called a particular solution.

The rest of the course focuses on systematic ways to find general and particular solutions for DE's.

For now, though, all we are expected to do is stuff like this:

Example: Show that every function in the 2-parameter family

$$y(x) = c_1 \sin(3x) + c_2 \cos(3x)$$

is a solution to $\dfrac{d^2y}{dx^2} + 9y = 0$.

Solution: Calculate two derivatives:

$$y'(x) = 3c_1 \cos(3x) - 3c_2 \sin(3x)$$

$$y''(x) = -9c_1 \sin(3x) - 9c_2 \cos(3x)$$

so plugging in we get

$$-9\cancel{c_1\sin(3x)} - 9\cancel{c_2\cos(3x)} + 9(\cancel{c_1\sin(3x)} + \cancel{c_2\cos(3x)})$$

$$=0.$$

So every function in the family is a solution.