§10.5 Taylor series.

So far we have learned two ways of getting an equation of the form

$$\sum_{n=0}^{\infty} a_n x^n = f(x).$$

① Start with the function $f(x)$, compute the Taylor series and verify that the remainders go to 0 as $n \to \infty$ to find where it converges.

② Start with a sum $\sum_{n=0}^{\infty} a_n x^n$, and by some tricks find a limit function $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and calculate convergence by $R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$

or $\lim_{n\to\infty} \frac{1}{\sqrt[n]{|a_n|}}$.

Question: By using these two different methods, is it possible to get two different formulas for the same function?

$$\sum_{n=0}^{\infty} a_n x^n = f(x) = \sum_{n=0}^{\infty} b_n x^n \quad ?$$

Answer (Theorem): No. There is only one power series expansion for a function $f(x)$.

and it is the Taylor (MacLaurin) series.

Here is an example of how we can use this:

<u>Example</u>: Find the MacLaurin series of $\dfrac{1}{4+5x}$.

<u>Solution</u>: We know one way of coming up with a power series for this function, and that's by using

$$\sum_{n=0}^{\infty} a x^n = \frac{a}{1-x}.$$

So if we write

$$\frac{1}{4+5x} = \frac{1}{4}\left(\frac{1}{1+\frac{5}{4}x}\right)$$

$$= \frac{1/4}{1+5/4\,x}.$$

Then we find

$$\frac{1}{4+5x} = \frac{1/4}{1+5/4\,x} = \sum_{n=0}^{\infty} \frac{1}{4}\left(\frac{-5x}{4}\right)^n$$

with convergence for $-1 < \frac{5x}{4} < 1$

ie. $-\frac{4}{5} < x < \frac{4}{5}$.

The theorem says that even though we found this series <u>without</u> using the Maclaurin series formula, it must actually be equal to the Maclaurin series. (Since fcn only has one power series at $c=0$).

So, what you can take away is this:
There is only one formula of the form
$\sum_{n=0}^{\infty} a_n (x-c)^n = f(x)$ that converges, for $R>0$ so no matter how you come up with the formula it'll agree with everyone else.

So here's a new way of coming up with formulas.

<u>Termwise integration/differentiation:</u>

If $f(x) = \sum_{n=0}^{\infty} a_n x^n$, for $-R < x < R$,

then
$$f'(x) = \sum_{n=0}^{\infty} \frac{d}{dx}(a_n x^n) = \sum_{n=0}^{\infty} n a_n x^{n-1}$$

for $-R < x < R$ and
$$\int f(x) = \sum_{n=0}^{\infty} \int a_n x^n = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1} + C.$$

the same formulas also hold when you have a shift of "c" in the power series.

<u>Example</u>: Last day I struggled to calculate the Taylor series of $\tan^{-1}(x) = f(x)$. However, now we can do:

$\tan^{-1}(x) = f(x)$

$\Rightarrow \quad f'(x) = \dfrac{1}{1+x^2} = \dfrac{1}{1-(-x^2)}$ } so this looks like $\dfrac{a}{1-x}$ with $a=1$, x replaced by $-x^2$.

So $f'(x) = \sum_{n=0}^{\infty} 1\cdot(-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$ for $-1 < -x^2 < 1$

i.e. $-1 < x < 1$.

But now to get a formula for our original function $\tan^{-1}(x) = f(x)$, we just integrate:

$$f'(x) = \frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 - \cdots$$

So

$$\int f'(x) = \int \frac{1}{1+x^2} = \int 1 - \int x^2 + \int x^4 - \int x^6 + \int x^8 - \cdots$$

$$\Rightarrow \tan^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}, \ +C, \text{ set } x=0 \text{ to get } C=0.$$

which is what we found last day, and $-1<x<1$ stays the same.

<u>Example:</u>

Here's yet <u>another</u> way of getting the Taylor series of $\ln(x)$.

If $f(x) = \ln(x)$, then $f'(x) = \frac{1}{x} = \frac{1}{1-(1-x)}$

So $f'(x) = \sum_{n=0}^{\infty} (1-x)^n$ for $-1 < 1-x < 1$

$\Rightarrow -2 < -x < 0$

$\Rightarrow 0 < x < 2.$

Now we want to change it to the form $(x-c)^n$ to apply our Taylor series theorems. So note that

$(1-x) = (-1)(x-1)$

so $(1-x)^n = (-1)^n (x-1)^n$

Therefore $\frac{1}{x} = 1-(x-1) + (x-1)^2 - (x-1)^3 + (x-1)^4 - \ldots$

$\Rightarrow \int \frac{1}{x} = x - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \ldots + C$

$\Rightarrow \ln|x| = x + \sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n}(x-1)^n + C$, setting $x=1$ gives

$0 = 1 + C$

$\Rightarrow C = -1$, so we can write

$\ln|x| = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(x-1)^n$ for $0 < x < 2$,

or just $\ln(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(x-1)^n$ for $0 < x < 2$

since $\ln|x| = \ln(x)$

for x between 0 and 2.

Example: What is the Maclaurin series of $f(x) = \dfrac{1}{(2-x)^3}$?

Solution: We know the ~~de~~ series of $\dfrac{1}{2-x}$,

it is:

$$\frac{1}{2-x} = \frac{\frac{1}{2}}{1-\frac{x}{2}} = \sum_{n=0}^{\infty} \frac{1}{2}\left(\frac{x}{2}\right)^n \qquad -1 < \frac{x}{2} < 1 \Rightarrow -2 < x < 2$$

We also know that we can go from $\dfrac{1}{2-x}$ to $\dfrac{1}{(2-x)^3}$ by differentiating twice (roughly). So:

$$\frac{1}{2-x} = \frac{1}{2}\sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n$$

$$\Rightarrow \frac{d}{dx}\frac{1}{2-x} = \frac{1}{2}\sum_{n=0}^{\infty} \frac{d}{dx}\left(\frac{x}{2}\right)^n$$

$$\Rightarrow \frac{+1}{(2-x)^2} = \frac{1}{2}\sum_{n=0}^{\infty} \frac{n x^{n-1}}{2^n}$$

$$\Rightarrow \frac{d}{dx}\frac{-1}{(2-x)^2} = \frac{1}{2}\sum_{n=0}^{\infty} \frac{d}{dx}\frac{n x^{n-1}}{2^n}$$

$$\Rightarrow \frac{+2}{(2-x)^3} = \frac{1}{2}\sum_{n=0}^{\infty} \frac{n(n-1)x^{n-2}}{2^n}$$

So $$\frac{1}{(2-x)^3} = \frac{1}{4}\sum_{n=0}^{\infty} \frac{n(n-1)x^{n-2}}{2^n} = \sum_{n=0}^{\infty} \frac{n(n-1)x^{n-2}}{2^{n+2}}$$

If you want to make this look like the textbook's answer, replace n with $n+1$. Then instead of starting at $n=0$ we start at $n=1$ and get

$$\frac{1}{(2-x)^3} = \sum_{n=1}^{\infty} \frac{n(n+1)x^{n-1}}{2^{n+3}} \quad \text{for } -2 < x < 2.$$

We can do other operations to get new power series from old ones, such as add them. (or subtract).

$$\sum_{n=0}^{\infty} a_n x^n \pm \sum_{n=0}^{\infty} b_n x^n = \sum_{n=0}^{\infty} (a_n \pm b_n) x^n$$

we can even multiply and divide them, formulas for which we get to find out next day.

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Basically, the lesson this class is:

You can add, subtract, multiply (and even divide) power series. If you start with two power series having radii of convergence R_1 and R_2, then the radius of convergence of their sum, difference or product is the smaller of R_1 and R_2.

Example: What is the power series expansion of

$$f(x) = \frac{3x}{x^2-2x-3} \ ?$$

Solution: Using partial fractions, we can write

$$\frac{3x}{x^2-2x-3} = \frac{3x}{(x-3)(x+1)} = \frac{A}{x-3} + \frac{B}{x+1}$$

and we solve $3x = A(x+1) + B(x-3)$

$$= Ax + A + Bx - 3B$$

$$\Rightarrow \begin{aligned} 3 &= A+B \\ 0 &= A-3B \end{aligned}$$

So $3 = 4B \Rightarrow B = 3/4$ and $A = \frac{9}{4}$.

So

$$\frac{3x}{x^2-2x-3} = \frac{9/4}{x-3} + \frac{3/4}{x+1}$$

$$= \frac{-1}{3}\left(\frac{9/4}{1-\frac{x}{3}}\right) + \frac{3/4}{1-(-x)}$$

$$= \frac{-3/4}{1-x/3} + \frac{3/4}{1-(-x)}$$

$$= \sum_{n=0}^{\infty} -3/4\left(\frac{+x}{3}\right)^n + \sum_{n=0}^{\infty} \frac{3}{4}(-x)^n$$

first series converges for $-1 < \frac{x}{3} < 1$, second for $-1 < x < 1$.

$$= \sum_{n=0}^{\infty} \frac{-x^n}{4 \cdot 3^{n-1}} + \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 3}{4} x^n$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 3^n - 1}{4 \cdot 3^{n-1}} x^n.$$

converges for the smaller of $-3 < x < 3$ and $-1 < x < 1$, so converges for $-1 < x < 1$.

Here's the theorem explaining what we just did:

Theorem: If $f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$ and $g(x) = \sum_{n=0}^{\infty} b_n (x-c)^n$, then

$$f(x) \pm g(x) = \sum_{n=0}^{\infty} (a_n \pm b_n)(x-c)^n$$

with convergence on the smaller of the two intervals of convergence.

__Example:__ What is the power series for
$$f(x) = \ln(x) + \frac{1}{3-x} \quad \text{centered at } x=1\,?$$

__Solution:__ The power series for $\ln(x)$ is
$$\ln(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(x-1)^n, \quad \text{for } 0<x<2.$$

The power series for $\frac{1}{3-x}$ __centered at 1__ is found by setting $y=x-1$ then finding the power series ~~for~~ in y at 0:
$$\frac{1}{3-x} = \frac{1}{2-(x-1)} = \frac{1/2}{1-y/2}. \quad \text{for}$$

So
$$\frac{1}{3-x} = \sum_{n=0}^{\infty} \frac{1}{2}\left(\frac{y}{2}\right)^n \quad \text{for } -1<\frac{y}{2}<1$$
$$\Rightarrow -2<y<2.$$

$$= \sum_{n=0}^{\infty} \frac{y^n}{2^{n+1}} \quad \text{for } -2<y<2.$$
$$= \sum_{n=0}^{\infty} \frac{(x-1)^n}{2^{n+1}} \quad \text{for } -2<x-1<2$$
$$\Rightarrow -1<x<3.$$

So
$$f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(x-1)^n + \sum_{n=0}^{\infty} \frac{1}{2^{n+1}}(x-1)^n$$
for $0<x<2$.

To combine, we take the $n=0$ term from the second sum out front:

$$\frac{1}{2} + \sum_{n=1}^{\infty} \left(\frac{(-1)^{n+1}}{n} + \frac{1}{2^{n+1}} \right) (x-1)^n = f(x) \text{ for } 0<x<2.$$

We can also multiply, the precise rule is:

<u>Theorem</u>: If $f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$ and $g(x) = \sum_{n=0}^{\infty} b_n (x-c)^n$,

then
$$f(x)g(x) = \left(\sum_{n=0}^{\infty} a_n (x-c)^n \right) \left(\sum_{n=0}^{\infty} b_n (x-c)^n \right)$$
$$= \sum_{n=0}^{\infty} d_n (x-c)^n$$

where $d_n = \sum_{i=0}^{n} a_i b_{n-i}$.

<u>In other words</u>: <u>multiply them like polynomials</u>!

<u>Example</u>: Calculate the first 3 terms in the power series expansion at 0 of $f(x) = \sin(x)\cos(x)$.

<u>Solution</u>: We multiply! Replacing $\sin(x)$ and $\cos(x)$ by their series at zero, we get

$$f(x) = \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \right) \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \right)$$

When we multiply this out, we get:

$$f(x) = x\cos(x) - \frac{x^3}{3!}\cos(x) + \frac{x^5}{5!}\cos(x), \text{ or}$$

$$= x\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots\right) - \frac{x^3}{3!}\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots\right) + \frac{x^5}{5!}\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots\right)$$

$$= \left(x - \frac{x^3}{2!} + \frac{x^5}{5!} - \cdots\right) + \left(\frac{-x^3}{3!} - \frac{x^5}{3!2!} + \frac{x^7}{3!4!}\right) + \left(\frac{x^5}{5!} - \frac{x^7}{5!2!} + \frac{x^9}{5!4!} - \cdots\right)$$

grouping like powers

$$= x - \left(\frac{1}{2!} + \frac{1}{3!}\right)x^3 + \left(\frac{1}{4!} + \frac{1}{2!3!} + \frac{1}{5!}\right)x^5 + O(x^7)$$

$$= x - \frac{2}{3}x^3 + \frac{2}{15}x^5 + O(x^7).$$

Division of power series:

It's done like long division of polynomials. We'll skip this, except for occasional simple examples.

Next, we add another "basic formula" to our repertoire. You may have seen before:

$$(a+b)^m = \sum_{n=0}^{m} \binom{m}{n} a^n b^{m-n}, \quad \text{where}$$

$$\binom{m}{n} = \frac{m!}{(m-n)!\,n!}, \quad \text{the binomial formula, here}$$

m is an integer.

we can extend this to non-integer values of m by using power series:

<u>Theorem</u>: Let $f(x) = (1+x)^m$, m any real number. Then the power series expansion of $f(x)$ at zero is

$$f(x) = \sum_{n=1}^{\infty} \frac{m(m-1)(m-2)\dots(m-n+1)}{n!} x^n.$$

Where this converges depends on the value of m, but we can say that no matter the value of m, it converges for $-1 < x < 1$.

(For some m values it converges ~~for~~ on a bigger interval, but this is the only interval where it converges regardless of m).

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At the end of last class we saw the binomial expansion

$$(x+y)^m = \sum_{n=0}^{m} \binom{m}{n} a^n b^{m-n}$$ which works for any positive integer m.

Here $\binom{m}{n} = \dfrac{m!}{(m-n)!\,n!}$

We can generalize this to a power series of the function $f(x) = (1+x)^m$ where m is any number:

$$(1+x)^m = 1 + \sum_{n=1}^{\infty} \frac{m(m-1)(m-2)\cdots(m-n+1)}{n!} x^n$$

with convergence for $-1 < x < 1$ and convergence everywhere when m is a positive integer.

Example (From tutorial)

Use the binomial expansion to find the Taylor series of $\dfrac{1}{(2-x)^3}$ at $x=0$.

Solution: The binomial formula applies to things of the form $(1+x)^m$, we have $(2-x)^{-3}$.

So we rewrite:

$$(2-x)^{-3} = \left(2\left(1+\left(\tfrac{-x}{2}\right)\right)\right)^{-3}$$

$$= \frac{1}{2^3}\left(1+\left(\tfrac{-x}{2}\right)\right)^{-3}$$

Then apply the binomial formula with $m=-3$ and x replaced by $\frac{-x}{2}$:

$$\frac{1}{(2-x)^3} = \frac{1}{2^3}\left(1+\sum_{n=1}^{\infty} \frac{(-3)(-4)(-5)\cdots(-3-n+1)}{n!}\left(\frac{-x}{2}\right)^n\right)$$

$$= \frac{1}{2^3}\left(1+\sum_{n=1}^{\infty} \frac{\cancel{(-1)^n}(3)(4)(5)\cdots(n+2)}{n!}\frac{\cancel{(-1)^n}}{2^n}x^n\right)$$

$$= \frac{1}{2^3}\left(1+\sum_{n=1}^{\infty} \frac{\cancel{(3)}\cancel{(4)}\cancel{(5)}\cdots(n+2)}{1\cdot 2\cdot \cancel{3}\cdots\cancel{(n-1)}\cancel{(n)}}\cdot\frac{1}{2^n}x^n\right)$$

$$= \frac{1}{2^3}\left(1+\sum_{n=1}^{\infty} \frac{(n+1)(n+2)}{2^{n+1}}x^n\right).$$

$$= \frac{1}{8}+\sum_{n=1}^{\infty} \frac{(n+1)(n+2)}{2^{n+4}}x^n$$

Example: Find the Maclaurin series of $\sin^{-1}(x)$.

Solution: The dererivative of $\sin^{-1}(x)$ is $\frac{1}{\sqrt{1-x^2}}$, or $(1-x^2)^{-1/2}$. So applying the binomial theorem with $m=-\frac{1}{2}$ and x replaced by $-x^2$ gives:

$$\frac{1}{\sqrt{1-x^2}} = 1+\sum_{n=1}^{\infty} \frac{(-\frac{1}{2})(-\frac{1}{2}-1)\cdots(-\frac{1}{2}-n+1)}{n!}(-x^2)^n$$

$$= 1+\sum_{n=1}^{\infty} \frac{\cancel{(-1)^n}(\frac{1}{2})(\frac{3}{2})(\frac{5}{2})\cdots(\frac{2n-1}{2})\cancel{(-1)^n}}{n!} x^{2n}$$

$$= 1+\sum_{n=1}^{\infty} \frac{1\cdot 3\cdot 5\cdot \ldots \cdot(2n-1)}{2^n\, n!} x^{2n}$$

Now we integrate both sides to get back to $\sin^{-1}(x)$:

$$\sin^{-1}(x) = x+\sum_{n=1}^{\infty} \frac{1\cdot 3\cdot 5\cdot \ldots \cdot(2n-1)}{2^n\, n!\,(2n+1)} x^{2n+1} + C.$$

If we plug in $x=0$ to find C, we get $0=0+0+C$, so $C=0$. Therefore

$$\sin^{-1}(x) = x+\sum_{n=1}^{\infty} \frac{1\cdot 3\cdot 5\ldots(2n-1)}{2^n\, n!\,(2n+1)} x^{2n+1}.$$

The book does some extra stuff to get to

$$\sin^{-1}(x) = \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(2n+1)(n!)^2} x^{2n+1},$$ don't worry too much about that.

Example: Find the Taylor series of

$f(x) = (1-2x)^{+1/3}$ about $x=1$.

Solution: We want to make a substitution $y = x-1$ and then find a series in y about 0.

So with $y = x-1$

$$(1-2x)^{1/3} = (1-2(x-1)-2)^{1/3}$$

$$= (-1-2y)^{1/3} = (-1)^{1/3}(1+2y)^{1/3}. \quad \text{But } (-1)^{1/3} = -1.$$

$$= -(1+2y)^{1/3}$$

Now the binomial theorem says

$$-(1+2y)^{1/3} = -1 - \sum_{n=1}^{\infty} \frac{\left(\frac{1}{3}\right)\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)\cdots\underbrace{\left(\frac{1}{3}-n+1\right)}_{\frac{4-3n}{3}}}{n!}(2y)^n$$

$$= -1 - \sum_{n=1}^{\infty} \frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)\cdots\left(\frac{4}{3}-n\right)}{n!} 2^n y^n$$

$$= -1 - \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\, 2\cdot 5\cdot 8\cdot \ldots \cdot(3n-4)}{3^n \cdot n!} \cdot 2^n y^n$$

$$= -1 - \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n (-1)^{n-1} \frac{2\cdot 5\cdot 8\ldots(3n-4)}{n!}(x-1)^n$$

with convergence for $-1 < y < 1$

$$\Rightarrow -1 < x-1 < 1 \Rightarrow 0 < x < 2.$$

<u>Example</u>: Find the sum and interval of convergence of

$$\sum_{n=0}^{\infty} \left(\frac{n+1}{n+2}\right) x^n$$

<u>Solution</u>: Whatever the sum is, call it $S(x)$, we'll solve for $S(x)$. Integrating x^n will give us some cancelation:

$$\int S(x)\,dx = \sum_{n=0}^{\infty} \left(\frac{n+1}{n+2}\right) \int x^n dx$$

$$= \sum_{n=0}^{\infty} \frac{1}{n+2} \cdot \frac{\cancel{n+1}}{\cancel{n+1}} x^{n+1} + C$$

Now we have something close to the formula:

$$\ln(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (x-1)^n \Rightarrow \ln(1-x) = \sum_{n=1}^{\infty} \frac{(-1)}{n} x^n$$

since we've got $n+2$ in the denominator. Set $m = n+2$,

Then

$$\int S(x)\,dx = \sum_{m=2}^{\infty} \frac{1}{m} x^{m-1} + C$$

$$= \frac{1}{x} \left(\sum_{m=2}^{\infty} \frac{1}{m} x^m \right) + C$$

this will become $\ln(1-x)$!

$$= \frac{1}{x} \left(x + \sum_{n=1}^{\infty} \frac{(-1)}{n} x^n \right) + C$$

$$= \frac{1}{x} \left(x + \ln(1-x) \right) + C$$

$$= 1 + \frac{\ln(1-x)}{x} + C$$

So, to get from

$$\int s(x)\,dx = 1 + \frac{\ln(1-x)}{x} + C.$$

To a formula for $s(x)$, differentiate! $s(x) = \frac{d}{dx}\left(\frac{\ln(1-x)}{x}\right)$, or

$$s(x) = \frac{1}{(x-1)x} - \frac{\ln(1-x)}{x^2}.$$

The interval of convergence is determined by the formula

$$\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}$$ that we used,

which converges for $0 < 1-x < 2$

$$\Rightarrow -1 < -x < 1$$

$$\Rightarrow -1 < x < 1.$$