January 19 Math 2132

We saw that approximating a function with Taylor polynomials and taking $c=0$ gives the Maclaurin series of that function.

E.g.
$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

When $c \neq 0$ we get the Taylor series, eg

$$\ln(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 + \cdots$$

This is just a difference of terminology.

In general, a sum of the form
$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$
is called a power series.

Eg. $$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$$

or $$\sum_{n=1}^{\infty} \frac{x^n}{2^n n!} = 1 + \frac{x}{2\cdot(1)} + \frac{x^2}{2^2 \cdot 2!} + \cdots$$

A power series may or may not converge to some function $f(x)$. Whether or not

$\sum_{n=0}^{\infty} a_n x^n = f(x)$ often depends on the particular x values you consider. When there's an interval I where $\sum_{n=0}^{\infty} a_n x^n = f(x)$ works, and it doesn't work outside of I, we call I the <u>interval of convergence</u>.

One way of coming up with a power series is to take a function $f(x)$ and calculate its MacLaurin series. We could also be given a power series and then be asked about its limit.

<u>E.g.</u> Consider the power series

$$\sum_{n=0}^{\infty} a x^n$$

This is called a <u>geometric series</u>, because every term after the first is obtained by multiplying the previous term by a common ratio x.

Recall that if $\sum_{n=0}^{\infty} a_n x^n = f(x)$ for some function $f(x)$,

then this is shorthand for saying

$$\lim_{n\to\infty} P_n(x) = f(x)$$

where $P_n(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$.

So in our case where $a_n = a$ is constant for all n, we find the limit function $f(x)$ by doing

$$\lim_{n\to\infty} (a + ax + ax^2 + ax^3 + \dots + ax^n) = f(x),$$ where the limit exists.

To take this limit, we rewrite $P_n(x)$:

$$P_n(x) = a + \cancel{ax} + \cancel{ax^2} + \cancel{ax^3} + \dots + \cancel{ax^n}$$

$$\Rightarrow \; xP_n(x) = \cancel{ax} + \cancel{ax^2} + \cancel{ax^3} + \cancel{ax^4} + \dots + ax^{n+1}$$

So $P_n(x) - xP_n(x) = a - ax^{n+1}$

$$\Rightarrow P_n(x) = \frac{a(1-x^{n+1})}{(1-x)} \quad (x \neq 1).$$

Now the limit $\lim_{n\to\infty} P_n(x)$ is easier:

$$\lim_{n\to\infty} P_n(x) = \frac{a(1 - \lim_{n\to\infty} x^{n+1})}{1-x} = \begin{cases} \frac{a}{1-x} & \text{if } |x|<1 \\ \text{does not exist otherwise} \end{cases}$$

because $\lim_{n\to\infty} x^{n+1}$ DNE if $|x|>1$.

Therefore we can write

$$\sum_{n=0}^{\infty} ax^n = \frac{a}{1-x} \quad \text{if } |x| < 1$$

END EXAMPLE

A <u>mathematically unsound</u> way of remembering this is:

Suppose $a + ax + ax^2 + ax^3 + ax^4 + \dots = S.$

multiply both sides by $(1-x)$

$$a(1-x) + ax(1-x) + ax^2(1-x) + \dots = S(1-x)$$

$$\Rightarrow a - ax + ax - ax^2 + ax^2 - ax^3 + \dots = S(1-x)$$

$$\Rightarrow a = S(1-x)$$

$$\Rightarrow S = \frac{a}{1-x}$$

The reason this is "mathematically unsound" is because our notation

$$a + ax + ax^2 + ax^3 + ax^4 + \dots \text{ etc} = S$$

doesn't mean "add infinitely many things together and get S"

It is a re-writing of the limit

$$\lim_{n\to\infty} P_n(x) = S.$$

It just happens to give the right answer in this case, so it's a good way of remembering.

In general, crazy things can happen if you treate an infinite sum "like a sum" instead of a limite.

Example: We saw

$$\ln(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}(x-1)^k \text{ for } 0 < x \le 2.$$

$$\text{So } \ln(2) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}(1) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots$$

But look what happens if we just treat this equation like it's "regular addition":

$$\ln(2) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$$

$$= \left(1 - \frac{1}{2}\right) - \frac{1}{4} + \left(\frac{1}{3} - \frac{1}{6}\right) - \frac{1}{8} + \left(\frac{1}{5} - \frac{1}{10}\right) \cdots$$ rearrange the order...

$$= \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} \cdots$$

$$= \frac{1}{2}\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} \cdots\right) = \frac{1}{2}\ln(2).$$

This happens because we're actually dealing with <u>limits</u> and not sums.

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Last day we saw that

$$\sum_{n=0}^{\infty} ax^n = a + ax + ax^2 + \dots = \frac{a}{1-x} \quad \text{if } |x|<1.$$

Before this, we saw the MacLaurin series for $\sin(x)$, which is

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots = \sin(x)$$

and it converges for all x.

Finally, we could make a power series that almost never converges, for example:

$$\sum_{n=0}^{\infty} n!\, x^n = 1 + x + 2!\,x^2 + 3!\,x^3 + \dots$$

this converges only if $x=0$. It turns out these are the only possibilities:

Theorem: Given a power series $\sum_{n=1}^{\infty} a_n x^n$, it converges in one of the following 3 ways:

1. Only for $x=0$,
2. Only for $|x|<R$, R some real number
3. For all values of x.

'R' in case 2 is called the 'radius of convergence'.
In cases (1) and (3) above, we say the radius is $R=0$ and $R=\infty$ respectively.

To find R, we have a theorem:

<u>Theorem</u>: The radius of convergence of the power series $\sum_{n=0}^{\infty} a_n x^n$ is given by

$$R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| \quad \text{or} \quad R = \lim_{n\to\infty} \frac{1}{\sqrt[n]{|a_n|}}$$

provided that either limit exists or is infinity.

<u>Example</u>: Find the interval of convergence for

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n\, 5^{2n}} x^n .$$

<u>Solution</u>: Here,

$$R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \left| \frac{\frac{(-1)^n}{n\,5^{2n}}}{\frac{(-1)^{n+1}}{(n+1)\,5^{2n+2}}} \right| = \lim_{n\to\infty} \left| \frac{(-1)^n (n+1) 5^{2n} \cdot 5^2}{(-1)^{n+1} \quad n \cdot 5^{2n}} \right|$$

$$= \lim_{n\to\infty} \frac{(n+1)}{n} \cdot 5^2 = 25.$$

So the interval of convergence is $(-25, 25)$ because we are in case ② above; $R = 25$ is the radius of convergence.

Example: The Maclaurin series for $\cos(x)$ is found to be:

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

For what x is this formula valid?

Solution: We can use the theorem to get R, first set $y = x^2$.

We find

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n\to\infty} \left| \frac{\frac{(-1)^n}{(2n)!}}{\frac{(-1)^{n+1}}{(2(n+1))!}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(2(n+1))!}{(2n)!} \right|$$

$$= \lim_{n\to\infty} \left| \frac{\cancel{1\cdot 2\cdot 3\cdot \ldots \cdot 2n}\cdot(2n+1)\cdot(2n+2)}{\cancel{1\cdot 2\cdot 3\cdot \ldots \cdot 2n}} \right|$$

$$= \lim_{n\to\infty} (2n+1)(2n+2) = \infty.$$ So $R = \infty$, and the formula is valid for all x (ie. the series converges everywhere).

Example: We found that

$$\ln(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (x-1)^n,$$ but this is not a power series so our theorem doesn't apply without changing variables.

we replace x with $x+1$ to get:

$$\ln(x+1) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} ((x+1)-1)^n = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n.$$

Now the right hand side is a power series, so we can calculate its radius! We find

$$R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \left|\frac{\frac{(-1)^{n+1}}{n}}{\frac{(-1)^{n+2}}{n+1}}\right| = \lim_{n\to\infty} \left|\frac{n+1}{n}\right| = 1.$$

So the formula $\ln(x+1) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n$ is true when $-1 < x < 1$.

Now, we can shift back to our original formula by replacing x with $x-1$:

$$\ln((x-1)+1) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (x-1)^n$$

$$\ln((x-1)+1) = \ln(x)$$

is true when $-1 < x-1 < 1$

ie $0 < x < 2$. (add 1).

This is what we've been saying all along.

Example: Find the radius of convergence of

$$\sum_{n=1}^{\infty} \frac{1}{n^2 2^n} x^{2n+1}.$$

Solution: Here, even powers of x do not appear in the series. So, this means $a_n = 0$ when n is even, and $\frac{a_n}{a_{n+1}}$ will give division by zero whenever $n+1$ is even.

So to fix the problem, we rewrite:

$$\sum_{n=1}^{\infty} \frac{1}{n^2 2^n} x^{2n+1} = x \sum_{n=1}^{\infty} \frac{1}{n^2 2^n} (x^2)^n \quad \text{set } y = x^2$$

$$= \sqrt{y} \sum_{n=1}^{\infty} \frac{1}{n^2 2^n} y^n .$$

Now we apply the theorem:

$$\lim_{n\to\infty} \left| \frac{\frac{1}{n^2 2^n}}{\frac{1}{(n+1)^2 2^{n+1}}} \right| = \lim_{n\to\infty} 2 \left(\frac{n+1}{n} \right)^2 = 2,$$ this gives convergence of the series in y. The series in x is related by $x = \pm\sqrt{y}$, so for our original series $R = \sqrt{2}$.

Example: Find R for $\sum_{n=1}^{\infty} \frac{1}{n^n} x^n$.

Solution: Here, we use the other limit:

$$R = \lim_{n\to\infty} \frac{1}{\sqrt[n]{|a_n|}} = \lim_{n\to\infty} \frac{1}{\sqrt[n]{1/n^n}} = \lim_{n\to\infty} \frac{1}{1/n} = \infty$$

So this series has $R = \infty$ and converges everywhere

§10.4 Continued.. MATH 2132 Jan 23.

Last day, we learned how to calculate the radius of convergence R of a power series $\sum a_n x^n$.
Then we did one example of a series like $\sum a_n (x-c)^n$, which you can analyze by "shifting" to get $\sum a_n x^n$, then you "shift back".

This always works in general:
The radius of convergence R of a 'shifted' power series $\sum a_n (x-c)^n$ can be computed in the same way as before. However you get convergence when x is within R of c, not 0. Ie.

converges for

$c - R < x < c + R$ insted of $-R < x < R$.

Example: Where does $\sum_{n=1}^{\infty} \frac{1}{5^n}(x-2)^n$ converge?

Solution: From our theorem last day.

$$R = \lim_{n\to\infty} \left| \frac{1}{\sqrt[n]{|5^n|}} \right| = \lim_{n\to\infty} \frac{1}{5} = \frac{1}{5}.$$

and then because of the shift, the series converges for x with $2-\frac{1}{5} < x < 2+\frac{1}{5}$.

Now knowing how to find <u>where</u> a sequence converges, we want to develop methods to say <u>what</u> it converges to. E.g.

$$\sum_{n=0}^{\infty} a x^n = \frac{a}{1-x}$$ or one we haven't seen

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

One way to find the limit is to manipulate the series to give the series a familiar form:

<u>Example</u>: Find the sum of $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2^n} x^n$.

<u>Solution</u>: We can regroup terms to make this look like $\sum_{n=0}^{\infty} a x^n$, which has sum $\frac{a}{1-x}$:

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2^n} x^n = \sum_{n=0}^{\infty} (-1) \left(\frac{-x}{2}\right)^n$$

all things with a power of n go here

now this is like $\sum a x^n$, with $a = (-1)$ and x replaced by $\frac{-x}{2}$, so $\sum_{n=0}^{\infty} ax^n = \frac{a}{1-x}$

$$\text{becomes } \sum_{n=0}^{\infty} (-1)\left(\frac{-x}{2}\right)^n = \frac{(-1)}{1-\left(\frac{-x}{2}\right)}$$

$$= \frac{-2}{2+x} \quad \left(\begin{array}{c}\text{mult top}\\ \text{and bottom}\\ \text{by } 2\end{array}\right)$$

Our original series converged for $-1 < x < 1$, but we replaced x with $\frac{-x}{2}$ so we get convergence for $-1 < \frac{-x}{2} < 1 \Rightarrow -2 < -x < 2$

$$\Rightarrow -2 < x < 2$$

Example: #34

Find the sum of the power series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} x^n$.

Solution: This looks like $\sum_{n=1}^{\infty} \frac{x^n}{n!} = e^x$, but it alternates

so it's not that. We get:

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} x^n = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} \cdots$$

which is actually

$$\sum_{n=1}^{\infty} \frac{(-x)^n}{n!} = e^{-x}.$$

Example: Find the sum of the series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{3^{2n+1}(2n+1)!} x^{2n+2}.$$

Solution: The $(-1)^n$ on top and $(2n+1)!$ make me think of $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = \sin(x)$. So let's try to isolate the part of the terms which look like this, and see what is left:

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{3^{2n+1}(2n+1)!} x^{2n+2} = \sum_{n=0}^{\infty} \frac{x}{3^{2n+1}} \cdot \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

Aha! Bring the x out, and group the 3^{2n+1} and x^{2n+1} together since they now have a common power:

$$= x \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \left(\frac{x}{3}\right)^{2n+1}$$

$$= x \sin\left(\frac{x}{3}\right).$$

Example: Where does $\displaystyle\sum_{n=1}^{\infty} 2^{n+1}(x+2)^n$ converge, and what does it converge to?

Solution: We find $\displaystyle\lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \left|\frac{2^n}{2^{n+1}}\right| = \frac{1}{2}.$

this is a series centered at -2, so it converges for x between: $-2-\frac{1}{2} < x < -2+\frac{1}{2}$

ie. $-5/2 < x < -3/2$.

To find the limit, we can rewrite this as

$$\sum_{n=1}^{\infty} 2^{n+1}(x+2)^n = \sum_{n=1}^{\infty} 2\left(2^n(x+2)^n\right)$$

$$= \sum_{n=1}^{\infty} 2\left(2(x+2)\right)^n$$

$$= \sum_{n=1}^{\infty} 2(2x+4)^n.$$

This looks like a geometric series $\sum_{n=0} ax^n = \frac{a}{1-x}$, with x replaced by $2x+4$. However, it starts at $n=1$ instead of $n=0$! So we fix this by observing:

$$\sum_{n=1}^{\infty} 2(2x+4)^n = \left[\sum_{n=0}^{\infty} 2(2x+4)^n\right] - 2\underbrace{(2x+4)^0}_{1}$$

$$= \frac{2}{1-(2x+4)} - 2$$

$$= \frac{2}{-2x-3} - 2 = \frac{2-(-4x-6)}{-2x-3}$$

$$= \frac{4(x+2)}{2x+3}.$$

END EXAMPLE.

Example: Find the sum of $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}(2n)!} x^{4n+4}$.

Solution: The $(2n)!$ makes me think of $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = \cos(x$

So we'll try to isolate those terms:

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}(2n)!} x^{4n+4} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \cdot \frac{x^{2n+2}}{2^{2n}}$$

However at this point it looks bad unless we group everything that has a common power of $2n$, then those terms will be the argument of cosine:

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \frac{x^{2n} \cdot x^{2n}}{2^{2n}} \cdot x$$

$$= x \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \left(\frac{x^2}{2}\right)^{2n} = x \cos\left(\frac{x^2}{2}\right).$$

This series converges for all x, since $\cos(x) = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$ converges for all x.