Section 10.2 Jan 11.

Last day we saw the example of a sequence of functions

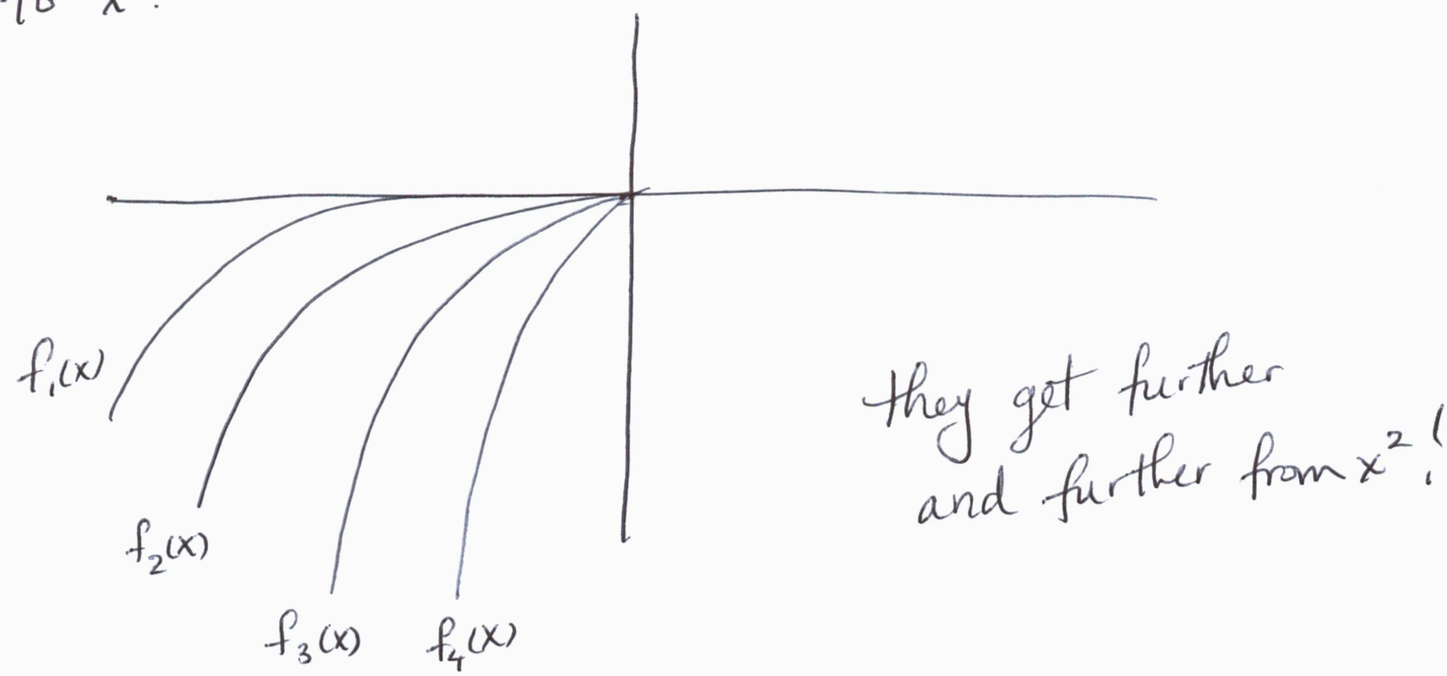
$$f_n(x) = x^2 + 10xe^{-nx}$$

which we said converged to $f(x) = x^2$: (graphically).

However, there is a subtle detail: What if we look at these functions and their behaviour when $x < 0$? Then

$$e^{-nx} \longrightarrow \infty \text{ as } x \longrightarrow -\infty \text{ and } n \to \infty.$$

So when x is negative, $f_n(x)$ do not get closer to x^2:



So when we write $\lim_{n\to\infty} f_n(x) = x^2$, we have to say where it is true. Our argument last day worked for $x \geq 0$, so we write

$$\lim_{n\to\infty} f_n(x) = x^2 \quad \text{for} \quad x \geq 0$$

In general, given $x_0 \in \mathbb{R}$ the numbers $f_n(x_0)$ are a sequence. Define $f(x_0) = \lim_{n\to\infty} f_n(x_0)$

Example: What is the limit of the functions

$$f_n(x) = \frac{\cos(nx)}{n} \quad ?$$

Solution: Without plotting, we can see that since $-1 \leq \cos(nx) \leq 1$ for all n and all x,

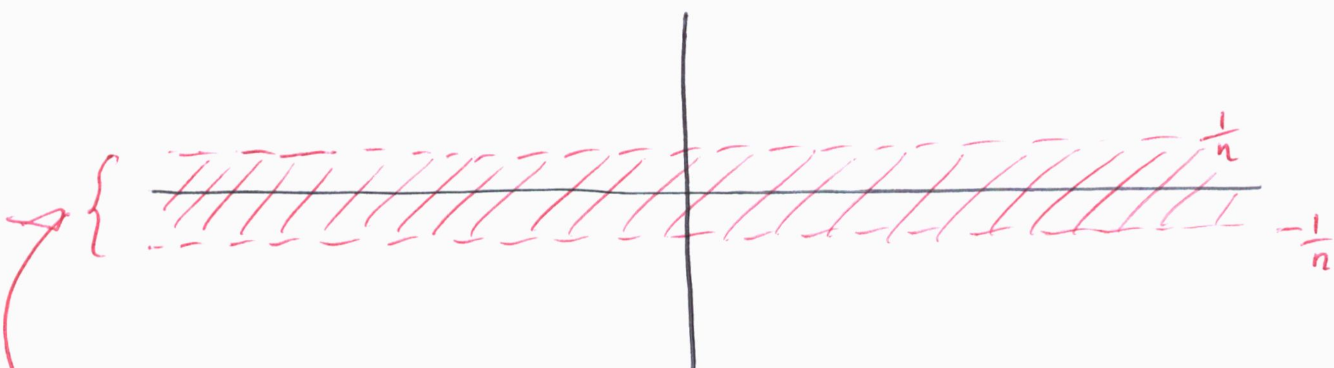
$$\lim_{n\to\infty} \frac{\cos(nx)}{n} \leq \lim_{n\to\infty} \frac{1}{n} = 0$$

and $$\lim_{n\to\infty} \frac{\cos(nx)}{n} \geq \lim_{n\to\infty} \frac{-1}{n} = 0$$

ie. $$0 \leq \lim_{n\to\infty} \frac{\cos(nx)}{n} \leq 0,$$ so the limit is zero for all values of x

In graphs, our inequality says that

$$-\frac{1}{n} \leq f_n(x) \leq \frac{1}{n},$$ so



our functions are within this red band for n large for all values of x!

An important example:

Let $f_n(x) = \dfrac{|x|^n}{n!}$. Show that the functions

$f_n(x)$ converge to the function $f(x)=0$.

Solution: If we want to make a formal argument using limits, this is a tricky one, so we'll do it together.

We can break up the product:

$$\frac{|x|^n}{n!} = \underbrace{\frac{|x|}{1} \cdot \frac{|x|}{2} \cdot \frac{|x|}{3} \cdot \ldots \cdot \frac{|x|}{n}}_{n \text{ factors.}}$$

Then if m is a number (integer) smaller than n (that we choose), we write

$$= \underbrace{\frac{|x|}{1} \cdot \frac{|x|}{2} \cdots \cdot \frac{|x|}{m-1} \cdot \frac{|x|}{m}}_{\text{call this "}M\text{".}} \cdot \ldots \cdot \frac{|x|}{n}.$$

Then
$$\frac{|x|^n}{n!} = M \cdot \frac{|x|}{m+1} \cdot \frac{|x|}{m+2} \cdot \ldots \cdot \frac{|x|}{n}$$

$$< M \cdot \frac{|x|}{m+1} \cdot \frac{|x|}{m+1} \cdot \ldots \cdot \frac{|x|}{m+1},$$

because replacing each denominator with a ~~to~~ smaller number makes the fraction bigger.

So we've arrived at

$$\frac{|x|^n}{n!} < M \cdot \left(\frac{|x|}{m+1}\right)^{n-m}$$

that's how many terms were in the product.

$$= M \underbrace{\left(\frac{|x|}{m+1}\right)^{-m}}_{\substack{\text{This part, when we're} \\ \text{taking a limit as } n\to\infty, \\ \text{is just a constant.}}} \cdot \left(\frac{|x|}{m+1}\right)^{n}.$$

So we need only concern ourselves with $\left(\frac{|x|}{m+1}\right)^n$.

But now a subtle point: If we're taking a limit $\lim\limits_{n\to\infty}\left(\frac{|x|}{m+1}\right)^n$, which x are we using? Is it $x=1$, -1, 100? Let's pick a particular x, say $x=5$, just to see how things work out there:

Then

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} f_n(5) \leq \lim_{n\to\infty} M \cdot \underbrace{\left(\frac{|5|}{m+1}\right)^{-m}}_{\text{constant}} \cdot \left(\frac{|5|}{m+1}\right)^{n}$$

$$= 0 \text{ if } \frac{|5|}{m+1} < 1.$$

But in this calculation, "m" is just an arbitrary number where we choose to break up the product

$$\frac{|x|}{1} \cdot \frac{|x|}{2} \cdot \cdots \cdot \frac{|x|}{n} .$$

So if we knew ahead of ~~th~~ time that we were going to focus on $x=5$, we could have chosen $m=6$, say, so that $\frac{|x|}{m+1} = \frac{5}{6+1} = \frac{5}{7} < 1$.

Then $\lim\limits_{n\to\infty} f_n(x) < \lim\limits_{n\to\infty} C\cdot\left(\frac{5}{7}\right)^n = 0$.

So, that's what we do: With some foresight, we suppose that for the particular x we're interested in, when we chose m we chose it so that $\frac{|x|}{m+1} < 1$.

Then

$$\lim_{n\to\infty} f_n(x) \leq \lim_{n\to\infty} C \left(\frac{|x|}{m+1}\right)^n = 0 .$$

Eg. Suppose we wanted to show $\lim\limits_{n\to\infty} f_n(x) = 0$ at $\underline{x=-10}$. Then choose $m=11$, say.

Then we can get

$$\frac{|x|^n}{n!} < M\cdot\left(\frac{|x|}{m+1}\right)^{-m} \cdot \left(\frac{|x|}{m+1}\right)^n \quad \text{at } \underline{x=-10}$$

and therefore $\left(\frac{|x|}{m+1}\right)^n = \left(\frac{10}{12}\right)^n \longrightarrow 0$ as $n \to \infty$.

So our limit argument works in the case $x=10$.

§10.3

Having learned about sequences of functions $\{f_n\}$ and their limits f, what is the use of a convergent sequence of functions $\{f_n\}$?

Well, if $\lim_{n\to\infty} f_n = f$, then we can think of each function $f_n(x)$ as an approximation of $f(x)$. The approximation gets better as we take larger and larger n, since $f_n(x)$ gets closer to $f(x)$.

So, if we have a particulary <u>ugly</u> function $f(x)$ that we need to work with, maybe we can approximate it by nice functions $f_n(x)$ converging to $f(x)$, then work with the nice functions instead.

Nicest possible functions: <u>Polynomials</u>. We now learn how to approximate by polynomials.

To show you how we'll go from a function $f(x)$ to an approximating sequence of polynomials $f_n(x)$, we need a theorem:

Rolle's Theorem:

Suppose $f(x)$ is a function that satisfies:

① $f(x)$ is continuous on $[a,b]$

② $f'(x)$ exists on (a,b)

③ $f(a) = f(b)$.

Then there is a number z_0 between a and b with $f'(z_0) = 0$.

So now, choose a function f defined on an interval $[c,x]$ with all of its derivatives $f'(x)$, $f''(x)$, etc its there.

Then make a new function

$$F(y) = f(x) - f(y) - \left(\frac{x-y}{x-c}\right)(f(x) - f(c)).$$

apply Rolle's theorem to it; we get z_0 with

$$0 = F'(z_0) = -f'(z_0) + \frac{1}{x-c}(f(x) - f(c))$$

Rearranging,

$$f(x) = f(c) + f'(z_0)(x-c).$$

We could start with an even crazier function:

$$F(y) = f(x) - f(y) - f'(y)(x-y) - \left(\frac{x-y}{x-c}\right)^2 [f(x) - f(c) - f'(c)(x-c)]$$

and apply Rolle's theorem to get a z_1 such that

$$0 = F'(z_1) = -f'(z_1) - f''(z_1)(x-z_1) + f'(z_1) + \frac{2(x-z_1)}{(x-c)^2}\left[f(x)-f(c)-f'(c)(x-c)\right]$$

and then solve for $f(x)$ to find

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(z_1)}{2}(x-c)^2.$$

Or, start with a function that is ~~3X~~ triple crazy:

$$F(y) = f(x) - f(y) - f'(y)(x-y) - f''(y)(x-y)^2$$
$$\qquad - \left(\frac{x-y}{x-c}\right)^3 \left(f(x) - f(c) - f'(c)(x-c) - f''(c)(x-c)^2\right)$$

and apply Rolle we get z_2 with

$$f(x) = f(c) + f'(c)(x-c) + f''(c)\frac{(x-c)^2}{2} + f'''(z_2)\frac{(x-c)^3}{3!}$$

Extending this pattern indefinitely we get

<u>Theorem 10.1</u>:

Suppose that $f(x)$ and its first n derivatives are continuous on $[c,x]$, and $f^{(n+1)}(x)$ is continuous on (c,x). Then there's a number z_n between c and x so that

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2}(x-c)^2 + \dots + \frac{f^{(n)}(c)}{n!}(x-c)^n$$
$$+ \frac{f^{(n+1)}(z_n)}{(n+1)!}(x-c)^{n+1}.$$

Since z_n is an "arbitrary" point in $[c, x]$, we don't really ever know in practice what the numerical value of $\frac{f^{(n+1)}(z_n)}{(n+1)!}(x-c)^{n+1}$ is.

So, we call it the nth Remainder, R_n.

Then our formula says that for each n, we have:

$$f(x) = f(c) + R_0$$

$$f(x) = f(c) + f'(c)(x-c) + R_1$$

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2}(x-c)^2 + R_2$$

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + R$$

We call the part before the remainder the Taylor polynomial of $f(x)$ about c:

So in general,

$f(x) =$ Taylor polynomial + remainder.

Example: What are the first few Taylor polynomials of $f(x) = e^x$ about $x=0$?

Solution: At $x=0$, $f'(x) = e^x$ $\Rightarrow f'(0) = e^0 = 1$ $\quad f''(x) = e^x$ $\Rightarrow f''(0) = e^0 = 1$, etc...

so the Taylor polynomials are

$P_0(x) = 1,$

$P_1(x) = 1 + x,$

$P_2(x) = 1 + x + \frac{x^2}{2}$

$P_3(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \dots$

In general, we add up terms of the form

$\frac{x^n}{n!}$.

<u>Example</u>: The Taylor polynomials of $\ln(x)$ about $x=1$ are

$P_0(x) = 0$

then using $\frac{d}{dx}\ln(x) = \frac{1}{x}$ and evaluating at $x=1$,

$P_1(x) = 0 + (1)(x-1) = x-1.$

then $\frac{d^2}{dx^2}\ln(x) = \frac{-1}{x^2}$ and evaluating at $x=1$:

$P_2(x) = (x-1) + \frac{(-1)}{2}(x-1)^2$

then $\frac{d^3}{dx^3}\ln(x) = \frac{2}{x^3}$, evaluate at $x=1$

$P_3(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3$, etc...

Point of all this:

$P_n(x)$ is a sequence of functions. We can show that the sequence will converge to $f(x)$ whenever the remainder R_n converge to zero!

Then, any operation we want to do on a function $f(x)$ can instead be done on $P_n(x)$ for n very large, and we'll get a reasonable approximation of the correct answer.

Section 10.3 c'td   January 16

Last day, we saw a formula that looked like:

$f(x)$ = Taylor polynomial + remainder.

The formula for the Taylor polynomial part is

$$P_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2}(x-c)^2 + \dots + \frac{f^{(n)}(c)}{n!}(x-c)^n$$

and the remainder is

$$R_n = \frac{f^{(n+1)}(z_n)}{(n+1)!}(x-c)^{n+1}.$$

If, for a particular function $f(x)$, we can show that the remainders go to 0 as $n \to \infty$, then the $P_n(x)$ approximate $f(x)$ very well. I.e.

$$\lim_{n\to\infty} P_n(x) = f(x),$$ the sequence of functions $\{P_n(x)\}$ converges.

Example:

We can show that if $f(x) = \sin(x)$, the Taylor polynomials converge to $f(x)$.

The Taylor polynomials of $f(x) = \sin(x)$ are, when $c=0$:

$P_0(x) = 0$

$P_1(x) = x$

$P_2(x) = x$

$P_3(x) = x - \dfrac{x^3}{3!}$

$P_4(x) = x - \dfrac{x^3}{3!}$

$P_5(x) = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!}$

$\vdots$ etc.

And in general, the remainder is (again, $c=0$):

$$R_n = \frac{f^{(n+1)}(z_n)}{(n+1)!}(x-0)^{n+1}$$

But what are the derivatives of $f(x) = \sin(x)$?

Well, $f^{(n+1)}(x)$ is one of $\pm\cos(x)$ or $\pm\sin(x)$.

So $f^{(n+1)}(z_n)$, no matter the value of z_n, is a number between -1 and 1:

$$-1 \le f^{(n+1)}(z_n) \le 1$$ multiply by $\dfrac{x^{n+1}}{(n+1)!}$

$$\Rightarrow \frac{-x^{n+1}}{(n+1)!} \leq \frac{f^{(n+1)}(z_n)}{(n+1)!} \cdot x^{n+1} \leq \frac{x^{n+1}}{(n+1)!}$$

which is the same as

$$0 \leq \left| \frac{f^{(n+1)}(z_n)\, x^{n+1}}{(n+1)!} \right| \leq \left| \frac{x^{n+1}}{(n+1)!} \right| = \frac{|x|^{n+1}}{(n+1)!}$$

or

$$0 \leq |R_n| \leq \frac{|x|^{n+1}}{(n+1)!}$$

So taking limits, and using the example from two classes ago, we get

$$\lim_{n\to\infty} |R_n| = 0 \quad \text{(by the squeeze theorem)}.$$

In other words:
When $f(x) = \sin(x)$ and $c = 0$, the remainders go to zero. So $\lim\limits_{n\to\infty} P_n(x) = \sin(x)$, where the formula for P_n is, for odd n:

$$P_n(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \pm \frac{x^n}{n!}$$

continue alternating signs

i.e. $$P_n(x) = \sum_{k=1}^{\left[\frac{n+1}{2}\right]} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$$

Instead of writing

$$\lim_{n\to\infty} P_n(x) = \sin(x),$$

we write

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} = \sin(x),$$

and this formula is called the <u>Maclaurin series</u> <u>for $\sin(x)$.</u>

<u>Example</u>: If we do the same procedure for $f(x) = \ln(x)$ and use $c=1$, we get polynomials:

$$P_n(x) = (x-1) - \frac{1}{2!}(x-1)^2 + \frac{2!}{3!}(x-1)^3 + \cdots$$

continue alternating signs

$$+ \frac{(-1)^{n+1}(n-1)!}{n!}(x-1)^n$$

and the formula for the remainder is

$$R_n = \frac{(-1)^n}{n+1}\left(\frac{x-1}{z_n}\right)^{n+1}.$$

In this case, we can only get

$$\lim_{n\to\infty} R_n = 0$$

to work if $0 < x \leq 2$.

(see example 10.11 in the book).

So, $\lim_{n\to\infty} P_n(x) = \ln(x)$ for $0 < x \le 2$.

or in other words

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}(k-1)!}{k!}(x-1)^k$$

$$:= \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}(x-1)^k = \ln(x) \text{ for } \frac{1}{2} \le x \le 2.$$

This is called the Taylor series of $\ln(x)$ at $x=1$.

Terminology: • If it's a sum with terms of the form $a_k x^k$, then $c=0$ and it's called a MacLaurin series.

• If it's a sum with terms $a_k(x-c)^k$, then it's called a Taylor series at c.

If the remainders only go to zero for some bit of the real line $[a,b]$, then $[a,b]$ is called the interval of convergence.

In the example of $f(x) = \ln(x)$, the interval of convergence is $0 < x \leq 2$, or $(0, 2]$.

In other words: For just about every purpose imagneable you can replace $\ln(x)$ with

$$\ln(x) = (x-1) - \frac{1}{2!}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \dots$$

(take as many terms as you need)

But your replacement formula will only approximate $\ln(x)$ if you restrict yourself to plugging in numbers between 0 and 2.

Example: If $f(x) = e^x$, then the remainder terms are all of the form (when $c=0$)

$$R_n = \frac{e^{z_n} x^{n+1}}{(n+1)!}$$

and no matter the value of z_n, these remainders go to zero. So we get

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \quad \text{for all } x.$$