Laplace Transforms of Dirac delta

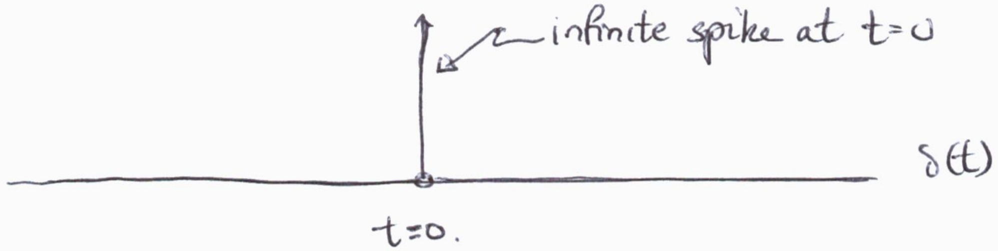
Last day we did an example that ended with a "tricky" inverse Laplace transform:

$$\mathcal{L}^{-1}\left\{\frac{s+1}{s+2}\right\} = \mathcal{L}^{-1}\left\{1 + \frac{(-1)}{s+2}\right\}$$

and the reason this was tricky is because we don't know how to do $\mathcal{L}^{-1}\{1\}$ (or some other "easy functions" are also missing from our table).

For this we need the Dirac delta "function".

It is a "function" $\delta(t)$ that you think of as being infinite at $t=0$ and 0 everywhere else:



So $\delta(t-a)$ has an "infinite spike" at $t=a$.

$\boxed{\text{Remark: } \delta(t) \text{ is not actually a function! It is called a distribution or a generalized function.}}$

For us, we only need to know 3 things in order to use it to solve DE's:

① $\delta(t-a) = 0$ if $t \neq a$.

(2) $\displaystyle\int_{a-r}^{a+r} \delta(t-a)\,dt = 1$, ie. integrating over the "infinite spike" of $\delta(t-a)$ gives 1.

(3) $$\int_{a-r}^{a+r} f(t)\,\delta(t-a)\,dt = f(a).$$

All of these properties tell us what we need to know about δ for Laplace transforms:

$$\mathcal{L}\{\delta(t-a)\} = e^{-as} \quad \text{so} \quad \mathcal{L}^{-1}\{e^{-as}\} = \delta(t-a)$$

and in particular $\mathcal{L}^{-1}\{1\} = \delta(t)$.

Remark: The Dirac delta is meant to model "instantaneous" forces, such as striking an object with a hammer or shorting a circuit.

(Paul's Notes)

Example: Solve the initial value problem

$$y'' + 2y' - 15y = 6\,\delta(t-9) \qquad y(0) = -5,\ y'(0) = 7.$$

Solution:

Step 1. Take $\mathcal{L}$ of both sides.

$$\mathcal{L}\{y''\} + 2\mathcal{L}\{y'\} - 15\mathcal{L}\{y\} = 6\,\mathcal{L}\{\delta(t-9)\}$$

$$\Rightarrow s^2 Y(s) - sy(0) - y'(0) + 2(sY(s) - y(0)) - 15Y(s) = 6e^{-9s}$$

$$\Rightarrow s^2 Y(s) + 5s - 7 + 2(sY(s)+5) - 15Y(s) = 6e^{-9s}$$

Step 2: Solve for $Y(s)$:

$$(s^2+2s-15)\,Y(s) = 6e^{-9s} - 5s + 7 - 10$$

$$\Rightarrow Y(s) = \frac{6e^{-9s}}{s^2+2s-15} - \frac{5s+3}{s^2+2s-15}$$

Step 3: Take inverse Laplace transforms.

The function $s^2+2s-15$ factors as $(s+5)(s-3)$

so

$$\frac{1}{s^2+2s-15} = \frac{A}{s+5} + \frac{B}{s-3}$$

$$\Rightarrow 1 = A(s-3) + B(s+5)$$

$$\Rightarrow A+B=0, \quad -3A+5B=1$$

$$\Rightarrow A = -\frac{1}{8}, \quad B = \frac{1}{8}$$

So

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+2s-15}\right\} = \frac{1}{8}e^{3t} - \frac{1}{8}e^{-5t}$$

and

$$① \quad \mathcal{L}^{-1}\left\{\frac{6e^{-9s}}{s^2+2s-15}\right\} = 6h(t-9)\left(\frac{1}{8}e^{3(t-9)} - \frac{1}{8}e^{-5(t-9)}\right)$$

On the other hand

$$\frac{5s+3}{s^2+2s-15} = \frac{A}{s+5} + \frac{B}{s-3}$$

$$\Rightarrow 5s+3 = A(s-3) + B(s+5)$$

$$\Rightarrow 5s+3 = (A+B)s + (5B-3A)$$

$\rightarrow$ $A+B=5$ and $5B-3A=3$, solve and get

$A = \frac{11}{4}$, $B = \frac{9}{4}$.

So 2

$$\mathcal{L}^{-1}\left\{\frac{5s+3}{s^2+2s-15}\right\} = \frac{11}{4}\mathcal{L}^{-1}\left\{\frac{1}{s+5}\right\} + \frac{9}{4}\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\}$$

$$= \frac{11}{4}e^{-5t} + \frac{9}{4}e^{3t}$$

Then

$$y(t) = 1 - 2$$

Note that the $\delta(t-9)$ results in a step function at $t=9$, ie $h(t-9)$.

Example: Solve the initial value problem

$$y'' + y = 4\delta(t-2\pi), \quad y(0) = y'(0) = 0.$$

Solution:

Step 1: Take $\mathcal{L}$ of both sides.

$$(s^2+1)Y(s) = 4e^{-2\pi s}$$

Step 2: Solve for $Y(s)$

$$Y(s) = 4\frac{e^{-2\pi s}}{s^2+1}$$

Step 3: Take inverse Laplace transforms:

$$y(t) = 4h(t-2\pi)\sin(t-2\pi)$$
$$= 4h(t-2\pi)\sin(t).$$

With the $\delta(t)$ function, we can now do inverse Laplace of functions we couldn't ~~before~~ before, like

$$\mathcal{L}^{-1}\left\{\frac{s+2}{s-1}\right\} = \mathcal{L}^{-1}\left\{1+\frac{3}{s-1}\right\} = \mathcal{L}^{-1}\{1\} + 3\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}$$
$$= \delta(t) + 3e^{+t}.$$

And in general, we can now do $\mathcal{L}^{-1}$ of functions $\frac{p(s)}{q(s)}$ where p, q are polynomials and the degree of p is greater than that of q.

Example: Calculate the inverse Laplace transform of

$$F(s) = \frac{2s^2+4s+5}{(s+1)(s+2)}$$

(Michael Adams text)

Solution: We do long division to change this problem into something familiar:

$$
\begin{array}{r}
2 \\
s^2+3s+2 \,\big)\overline{\,2s^2+4s+5\,} \\
-\underline{(2s^2+6s+4)} \\
-2s+1
\end{array}
$$

So this means we can rewrite this as:

$$\frac{2s^2+4s+5}{(s+1)(s+2)} = 2 + \frac{1-2s}{(s+1)(s+2)}$$

Then we use partial fractions on $\frac{1-2s}{(s+1)(s+2)}$ and proceed as we usually do:

$$\frac{1-2s}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}$$

$$\Rightarrow 1-2s = A(s+2)+B(s+1)$$

$$\Rightarrow A+B=-2, \quad 2A+B=1$$

$$\Rightarrow A=3, \quad B=-5.$$

So

$$\frac{2s^2+4s+5}{(s+1)(s+2)} = 2 + \frac{3}{s+1} - \frac{5}{s+2}$$

So

$$\mathcal{L}^{-1}\left\{\frac{s^2+4s+5}{(s+1)(s+2)}\right\} = 2\delta(t) + 3e^{-t} - 5e^{-2t}$$

So now we can take $\mathcal{L}^{-1}$ of all rational functions.

Namely, functions $\frac{p(s)}{q(s)}$. The algorithm is:

1. If the degree of p is greater than that of q, use long division to write
$$\frac{p(s)}{q(s)} = A + \frac{p'(s)}{q'(s)}$$
for some number/polynomial A, where degree p' < degree q'

2. Do partial fractions on $\frac{p'(s)}{q'(s)}$

3. Handle the pieces resulting from partial fractions by using reverse lookup and tables, except for pieces with a higher-power quadratic in the denominator. Then use convolution.

Last Lecture: One tricky example and some applications.

We saw how to take the Laplace transform of a periodic function with period p. The formula was

$$\mathcal{L}\{f(t)\} = \int_0^p e^{-st} f(t)\,dt \cdot \frac{1}{1-e^{-ps}}$$

These types of problems come up often in applications (especially circuits), so we'll do one.

Example: Solve

$$y'' + 4y = f(t), \quad y(0)=0, \quad y'(0)=0$$

where $f(t)$ is the periodic function

$$f(t) = \begin{cases} 1 \text{ if } 0 < t < 1 \\ 0 \text{ if } 1 < t < 2 \end{cases}$$

and $f(t)$ repeats, i.e. $f(t+2) = f(t)$ for all t.

Solution: As usual,

Step 1: Take Laplace transforms of both sides

$$\mathcal{L}\{y''\} + 4\mathcal{L}\{y\} = \mathcal{L}\{f(t)\}$$

$$\Rightarrow (s^2+4)Y(s) = \frac{1}{1-e^{-2s}} \int_0^2 e^{-st} \cdot f(t)\,dt$$

But $f(t) = 0$ from 1 to 2 so this becomes

$$\int_0^2 e^{-st} f(t)\,dt = \int_0^1 e^{-st}\,dt = \left[\frac{-1}{s} e^{-st}\right]_0^1$$

$$= \frac{1}{s} - \frac{e^{-s}}{s}.$$

So we have

$$(s^2+4)\,Y(s) = \frac{1}{1-e^{-2s}}\left(\frac{1}{s} - \frac{e^{-s}}{s}\right).$$

Step 2: Solve for $Y(s)$.

We get

$$Y(s) = \frac{1}{1-e^{-2s}} \cdot \frac{1-e^{-s}}{s} \cdot \frac{1}{s^2+4}.$$

But we can factor: $1-e^{-2s} = (1-e^{-s})(1+e^{-s})$

and then get $$Y(s) = \frac{1}{1+e^{-s}} \cdot \frac{1}{s} \cdot \frac{1}{s^2+4}.$$

Step 3: Take $\mathcal{L}^{-1}$.

Here is a trick we have not learned yet. First let's deal with the familiar piece

$$\frac{1}{s(s^2+4)} = \frac{A}{s} + \frac{Bs+C}{s^2+4}$$

and we get

$$1 = As^2 + 4A + Bs^2 + Cs$$

$$\Rightarrow 4A = 1, \quad C = 0, \quad A + B = 0$$

so $A = \frac{1}{4}$, $B = -\frac{1}{4}$. and

$$\frac{1}{s(s^2+4)} = \frac{1}{4} \cdot \frac{1}{s} - \frac{1}{4} \cdot \frac{s}{s^2+4}.$$

On the other hand, the term $\frac{1}{1+e^{-s}}$ is unlike anything we've looked at before... unless you think back to $\sum_{n=0}^{\infty} ax^n = \frac{a}{1-x}$, then you get

$$\frac{1}{1+e^{-s}} = \frac{1}{1-(-e^{-s})} = \sum_{n=0}^{\infty} (-1)^n (e^{-s})^n$$

So

$$Y(s) = \left(\frac{1}{s} - \frac{s}{s^2+4}\right) \cdot \frac{1}{4} \cdot \left(1 - e^{-s} + e^{-2s} - e^{-3s} + e^{-4s} - \ldots\right)$$

$$= \frac{1}{4} \sum_{n=0}^{\infty} \left[(-1)^n e^{-sn} \left(\frac{1}{s} - \frac{s}{s^2+4}\right)\right]$$

and we can actually take $\mathcal{L}^{-1}$ of each term individually! We get

$$\mathcal{L}^{-1}\left\{(-1)^n e^{-sn} \left(\frac{1}{s} - \frac{s}{s^2+4}\right)\right\}$$

$$= (-1)^n h(t-n)\left(1 - \cos(2(t-n))\right)$$

So we can do a term-wise inverse Laplace transform of $Y(s)$ and get

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{4}\sum_{n=0}^{\infty}(-1)^n e^{-sn}\left(\frac{1}{s}-\frac{s}{s^2+4}\right)\right\}$$

$$= \frac{1}{4}\sum_{n=0}^{\infty}\mathcal{L}^{-1}\left\{(-1)^n e^{-sn}\left(\frac{1}{s}-\frac{s}{s^2+4}\right)\right\}$$

$$= \cancel{\frac{(-1)^n}{4}\sum_{n=0}^{\infty}} = \frac{1}{4}\sum_{n=0}^{\infty}(-1)^n h(t-n)\left(1-\cos(2(t-n))\right)$$

Natural question: How useful is an answer like this?

Ans: Very useful. If you want to plug in a particular value of t, like $t=100$, say, then only the first 100 terms will be nonzero. Anything of the form $h(t-n)(1-\cos(2(t-n)))$ will be zero if $n>100$, because of the step function.

Last class, I tried to sell you on the $\delta(t)$ function in two ways:

- It is needed to compute $\mathcal{L}^{-1}\{1\}$.
- It is useful "in real life"

I owe you proof of the second claim.

Example: Recall that we can model a mass bouncing on a spring by a 2nd order linear DE with constant coefficients:

$$m y'' + \beta y' + ky = f(t),$$

where m = mass

β = drag

k = spring constant

and $f(t)$ = external forces applied.

If $m = 1$kg, $\beta = 8$ and $k = 15$, with the mass initially at a height of 1m above equilibrium, model its movement if $f(t) = 4\delta(t-2)$. I.e at time $t=2$ seconds there is a sudden impulse of 4N.

This amounts to solving

$$y'' + 8y' + 15y = 4\delta(t-2), \quad y(0) = 1, \quad y'(0) = 0.$$

and I want to verify that the solution looks like what our intuition says it should!

Step 1: Take $\mathcal{L}$ of both sides

$$(s^2+8s+15)\,Y(s) = 4\,\mathcal{L}\{\delta(t-2)\} + \underbrace{s+8}_{\text{these come from the initial condition}}$$

$$= 4e^{-2s} + s + 8$$

Step 2: Solve for $Y(s)$:

$$Y(s) = \frac{4e^{-2s}}{s^2+8s+15} + \frac{s+8}{s^2+8s+15}.$$

Step 3: Take $\mathcal{L}^{-1}$ to find $y(t)$.

Partial fractions.

$$\frac{s+8}{s^2+8s+15} = \frac{s+8}{(s+3)(s+5)} = \frac{A}{s+3} + \frac{B}{s+5}$$

$$\Rightarrow \quad s+8 = A(s+5) + B(s+3)$$

$$\Rightarrow \quad 1 = A+B \quad \text{and} \quad 8 = 5A+3B$$

$$\Rightarrow \quad A = \frac{5}{2}, \quad B = -\frac{3}{2}.$$

$$\Rightarrow \quad \frac{s+8}{(s+3)(s+5)} = \frac{5}{2}\left(\frac{1}{s+3}\right) - \frac{3}{2}\left(\frac{1}{s+5}\right).$$

Same for other piece:

$$\frac{4}{(s+3)(s+5)} = 2\left(\frac{1}{s+3}\right) - 2\left(\frac{1}{s+5}\right).$$

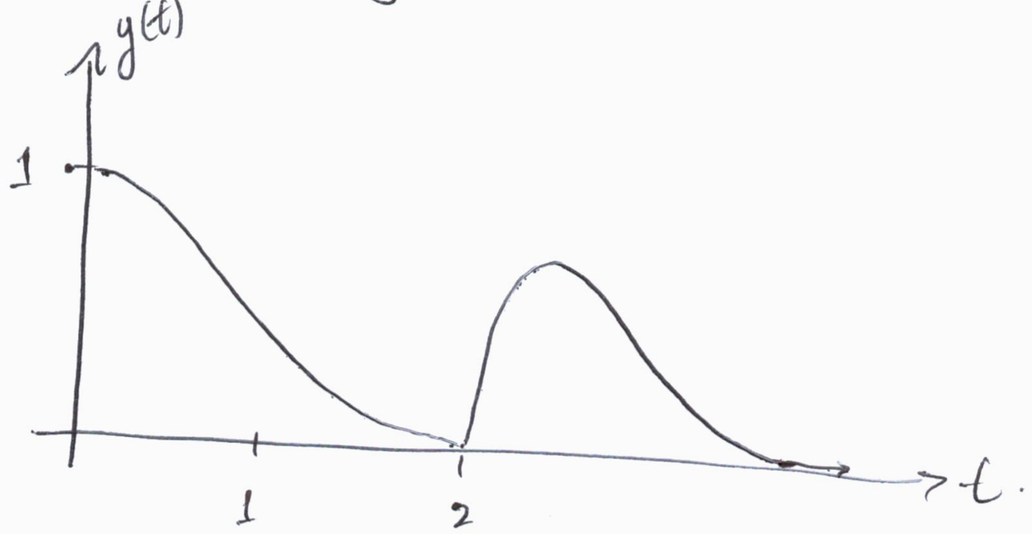
So then

$$Y(s) = \frac{5}{2}\left(\frac{1}{s+3}\right) - \frac{3}{2}\left(\frac{1}{s+5}\right) + e^{-2s}\left(2\left(\frac{1}{s+3}\right) - 2\left(\frac{1}{s+5}\right)\right)$$

So when we take inverse Laplace transforms:

$$y(t) = \frac{5}{2}e^{-3t} - \frac{3}{2}e^{-5t} + h(t-2)\left(2e^{-3(t-2)} - 2e^{-5(t-2)}\right).$$

If you plot this solution, the part that "turns on" at $t=2$ is due to the sudden force at ~~app~~ that time, and we get:



Note that there's no oscillation, the system is too damped for that (overdamped).

MATH 2132 Last class.

Final exam remarks: • The table of Laplace transforms that appears on the exam is literally cut and paste from the 2008 final exam (on dept website).

- The exam has 10 questions, 100pts total.
- $24/100$ pts on power series

  $24/100$ pts on first order / second order and linear ODE's

  $52/100$ pts on Laplace transforms.
- There is one long Laplace transform question that has multiple parts and it is worth $24/100$ pts

Office hours during exam period: None.
I will make appointments though, and if you want feel free to come as a group.

I will arrange a review before the exam where I'll do a final exam from a previous year. Possible days for this are: 13–17, 20 and 21 of April. The date and time will be decided by vote, if the class requests such a review.

Review of DE's material. (Chapter 15)

The order of a DE is the highest derivative that appears in the equation.

A general solution to a DE is a formula that contains arbitrary constants, and satisfies:

(*) for every particular solution to the DE, there is some choice of arbitrary constants that makes the general solution equal to your particular solution.

Types of DE's

① A DE is first-order linear if it can be written as $y' + P(x)y = Q(x)$.

In this case, the "integrating factor" is

$$\mu(x) = e^{\int P(x)\,dx}$$

and the solution is

$$y(x) = \frac{\int \mu(x)\,Q(x)\,dx + C}{\mu(x)}.$$

② A DE is separable if it can be written in the form

$$\frac{dy}{dx} = \frac{M(x)}{N(y)}$$ (ie, a function of x only divided by a function of y only)

In this case, you separate and integrate to find an implicit equation for y:

$$\int N(y)\,dy = \int M(x)\,dx$$

and then solve for y if you can.

③ Second-order DE's are <u>reducible to first-order</u> if either the independent variable or the dependent variable (Or both) is missing from the equation.

(i) If x, the independent variable, is missing:

Set $v = y'$ and $y'' = v\dfrac{dv}{dy}$, and solve the resulting first-order equation for v.

(ii) If y, the dependent variable, is missing:

Set $v = y'$, $v' = y''$ and solve the resulting first-order equation for v.

Regardless of the substitution made (either (i) or (ii)), after finding v you integrate to find y.

④ <u>Linear with constant coefficients</u>

In this case the DE looks like:

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1} \frac{dy}{dx} + a_n y = F(x).$$

(i) If $F(x) = 0$ then the general solution is called y_h and you find y_h by:

(a) Factor the complementary equation

$$a m^n + a_1 m^{n-1} + \dots + a_{n-1} m + a_n = 0$$

and find all the roots.

(b) Each root gives a solution according to the rules:

- Real roots r of multiplicity k give
$$(C_1 + C_2 x + \dots + C_k x^{k-1}) e^{rx}$$
- Complex roots $a \pm ib$ of multiplicity k give
$$e^{ax}\Big((C_1 + C_2 x + \dots + C_k x^{k-1}) \cos bx$$
$$+ (D_1 + D_2 x + \dots + D_k x^{k-1}) \sin bx\Big).$$

(c) Add together all the solutions from (b) to get y_h.

(ii) If $F(x) \neq 0$ then y_h is not the general solution, it is only part of the general solution and you must also find y_p. Then

$y_h + y_p$ is the general solution.

We only have a method for finding y_p when $F(x)$ consists of products/sums of polynomials, exponentials and sines/cosines.

Our method: Guess a general form for y_p, and plug y_p, y_p', y_p'', etc into the DE and solve for the coefficients.

Here is how you guess:

RHS of DE	Guess for y_p
$\alpha e^{\beta t}$	$Ae^{\beta t}$
$\alpha\cos(\beta t)$, $\alpha\sin(\beta t)$, or a sum of these	$A\cos\beta t + B\sin\beta t$
degree n polynomial	$A_n x^n + A_{n-1}x^{n-1} + \dots + A_1 x + A_0$
sum of the above	sum of corresponding guesses.
product of above	product of guesses, without doubling constants!

<u>Supplementary rule</u>: If any part of your guess for y_p overlaps with y_h, scale that part of your guess by powers of x until there is no overlap!

The reason that these approaches to solving linear homogeneous/nonhomongeneous DE's work is:

<u>Theorem</u> (Superposition principle).

If $y_1(x), y_2(x), \dots, y_n(x)$ are linearly independent solutions to an nth order ~~DE~~ homogeneous linear DE, then

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \dots + c_n y_n(x)$$

is a general solution.

<u>Recall</u>: Linearly independent means:

$y_1, y_2, \dots, y_n$ are linearly independent if

$c_1 y_1 + c_2 y_2 + \dots + c_n y_n = 0$ forces $c_1 = c_2 = c_3 = \dots = c_n = 0$.