§16.1 Laplace continued.

We saw the Laplace transform last day:

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt.$$

We also calculated that

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}, \quad s>a$$

and $\mathcal{L}\{t\} = \frac{1}{s^2}, \quad s>0.$

We ended by observing that

$$\mathcal{L}\{f(t)+g(t)\} = \int_0^\infty e^{-st}(f(t)+g(t))\,dt$$

$$= \int_0^\infty e^{-st} f(t)\,dt + \int_0^\infty e^{-st} g(t)\,dt$$

$$= \mathcal{L}\{f(t)\} + \mathcal{L}\{g(t)\}.$$

Also $\mathcal{L}\{cf(t)\} = \int_0^\infty e^{-st} c f(t)\,dt$

$$= c\int_0^\infty e^{-st} f(t)\,dt$$

$$= c\,\mathcal{L}\{f(t)\}.$$

So the Laplace transform is <u>linear</u>.

<u>Example</u>: Find the Laplace transform of

$$10e^{5t} + 6t.$$

<u>Solution</u>. Use linearity

Then

$$\mathcal{L}\{10e^{5t}+6t\} = 10\,\mathcal{L}\{e^{5t}\} + 6\,\mathcal{L}\{t\}$$

$$= \frac{10}{s-5} + \frac{6}{s^2}, \quad s>5 \text{ and } s>0$$

ie $\boxed{s>5}$.

<u>Example</u>: Find $\mathcal{L}\{\sin(at)\}$; $f(t)=\sin(at)$.

<u>Solution</u>: We calculate:

$$\mathcal{L}\{\sin(at)\} = \int_0^\infty e^{-st}\sin(at)\,dt = F(s)$$

to do this integral, we use the "double integration by parts" trick covered many lectures ago, and get:

$$F(s) = \frac{a}{s^2} - \frac{a^2}{s^2}F(s).$$

$$\Rightarrow F(s) = \frac{a}{s^2+a^2}$$

In general, we don't calculate Laplace transforms using the definition

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt.$$

Instead we look up answers in a table.

$f(t)$	$F(s) = \mathcal{L}\{f(t)\}$
e^{at}	$\dfrac{1}{s-a}$
t^n	$\dfrac{n!}{s^{n+1}}$
$\cos(at)$	$\dfrac{s}{s^2+a^2}$
$\sin(at)$	$\dfrac{a}{s^2+a^2}$
$e^{at}\sin(bt)$	$\dfrac{b}{(s-a)^2+b^2}$
$e^{at}\cos(bt)$	$\dfrac{s-a}{(s-a)^2+b^2}$.

+ many, many more that we'll cover.

However, once we start using tables this means we're also ready to do a "reverse lookup" and start taking inverse Laplace tranforms. For example $\mathcal{L}^{-1}\left\{\dfrac{1}{s^2+1}\right\} = \sin(t)$, because we can just look up the entry for $\sin(t)$ and go backwards.

Example: What is the inverse Laplace transform of $\dfrac{1}{2s^2+3s+1}$?

Solution: To make this look like something in the table, we need to either:

(i) Factor it, and then use partial fractions to write it as $\dfrac{A}{cs-a} + \dfrac{B}{ds-b}$

where its factors are $cs-a$ and $ds-b$.

(ii) Check that it doesn't factor, and then complete the square to get something like:

$$\frac{1}{(s-a)^2+b^2}.$$

We try factoring first:

$$2s^2+3s+1 = (s+1)(2s+1), \text{ it worked!}$$

So partial fractions:

$$\frac{1}{2s^2+3s+1} = \frac{A}{s+1} + \frac{B}{2s+1}$$

$$\Rightarrow 1 = 2As + A + Bs + B$$

$$\Rightarrow \left.\begin{array}{l} 2A+B=0 \\ A+B=1 \end{array}\right\} \Rightarrow B = -2A$$

so

$$A - 2A = 1$$

$$\Rightarrow A = -1 \text{ and } B = 2.$$

So $\dfrac{1}{2s^2+3s+1} = \dfrac{-1}{s+1} + \dfrac{2}{2s+1}$ $\qquad \dfrac{1}{s+\frac{1}{2}}$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{2s^2+3s+1}\right\} = \mathcal{L}^{-1}\left\{\frac{-1}{s+1}\right\} + \mathcal{L}^{-1}\left\{\frac{2}{2s+1}\right\}$$

$$= -\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s+\frac{1}{2}}\right\}$$

$$= -e^{-t} + e^{-\frac{1}{2}t},$$ each above is $\dfrac{1}{s-a}$.

<u>Example</u> Calculate $\mathcal{L}^{-1}$ of $\dfrac{s}{s^2-2s+5}$.

<u>Solution</u>: If it factors, we factor and use partial fractions. Otherwise, complete the square on bottom.

(i) Does it factor?

The deserimminant is b^2-4ac

$$= (-2)^2 - 4(5)(1)$$

$$= 4 - 20 = -16 < 0,$$

so it does not factor.

So, complete the square. We get

$$s^2 - 2s + 5 = (s-1)^2 + 2^2$$

(In general, $s^2+bs+c = (s+\frac{1}{2}b)^2 + k^2$, where k is a constant

So we have $\dfrac{s}{s^2-2s+5} = \dfrac{s}{(s-1)^2+2^2}$

We'd like to use

$$e^{at}\cos(bt) = \frac{s-a}{(s-a)^2+b^2}$$

so we need $s-1$ on top!

So we do:

$$\frac{s}{(s-1)^2+2^2} = \frac{s-1}{(s-1)^2+2^2} + \frac{1}{(s-1)^2+2^2}$$

This now needs to look like $\dfrac{b}{(s-a)^2+b^2}$ to use the sin formula.

$$= \frac{s-1}{(s-1)^2+2^2} + \frac{1}{2}\,\frac{2}{(s-1)^2+2^2}$$

So inverse Laplace is

$$e^{t}\cos(2t) + \tfrac{1}{2}e^{t}\sin(t).$$

§16.2

Algebraic properties of the Laplace transform.

We've introduced the Laplace transform, which changes $f(t)$ into $F(s)$ via

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt.$$

We also saw a basic table of Laplace transforms last day. Today, we study how "small changes" to $f(t)$ affect the output of the Laplace transform, $F(s)$. This will give us new formulas for our tables based upon making small changes to old formulas.

First small change: Multiply $f(t)$ by e^{at}.

If $f(t)$ is multiplied by e^{at}, here is what happens to the Laplace transform $F(s)$:

$$\mathcal{L}\{e^{at} f(t)\} = \int_0^\infty e^{-st} e^{at} f(t)\,dt$$

$$= \int_0^\infty e^{-(s-a)t} f(t)\,dt$$

But recall the formula for $F(s)$ is

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt,$$ so what we've written

above is this formula with s replaced by $s-a$.

Therefore:

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a)$$

and equivalently $\mathcal{L}^{-1}\{F(s-a)\} = e^{at}f(t)$.

<u>Example</u>: Calculate $\mathcal{L}\{e^{at}t^n\}$.

<u>Solution</u>: From the table we know that

$\mathcal{L}\{t^n\} = \dfrac{n!}{s^{n+1}}$. So multiplying by e^{at} has the effect of shifting by a, and we get

$$\mathcal{L}\{e^{at}t^n\} = \frac{n!}{(s-a)^{n+1}}.$$

<u>Example</u>: Calculate $\mathcal{L}\{e^{at}\sin(bt)\}$ without using tables.

<u>Solution</u>: We already saw an outline of how to compute

$$F(s) = \mathcal{L}\{\sin(bt)\} = \int_0^\infty e^{-st}\sin(bt)\,dt$$

$$\Rightarrow \quad F(s) = \frac{b}{s^2+b^2}.$$

Then multiplying by e^{at} has the effect of shifting:

$$\mathcal{L}\{e^{at}\sin(bt)\} = F(s-a) = \frac{b}{(s-a)^2+b^2},$$

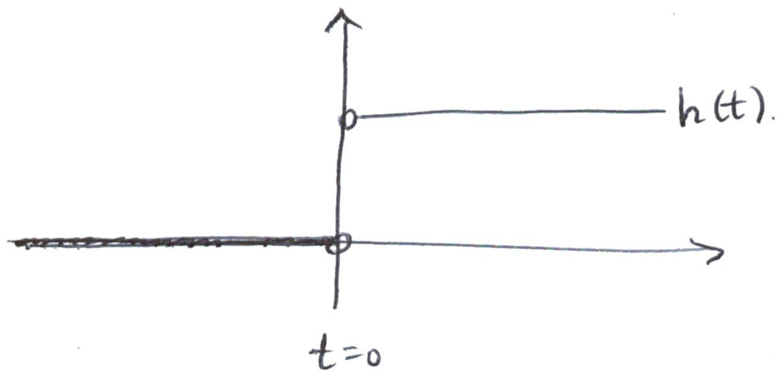
which agrees with the table.

<u>Second "small change"</u>.

We'll multiply by the function $h(t)$, called the unit step function:

$$h(t) = \begin{cases} 0, & t<0 \\ 1, & t \geqslant 0 \end{cases}$$

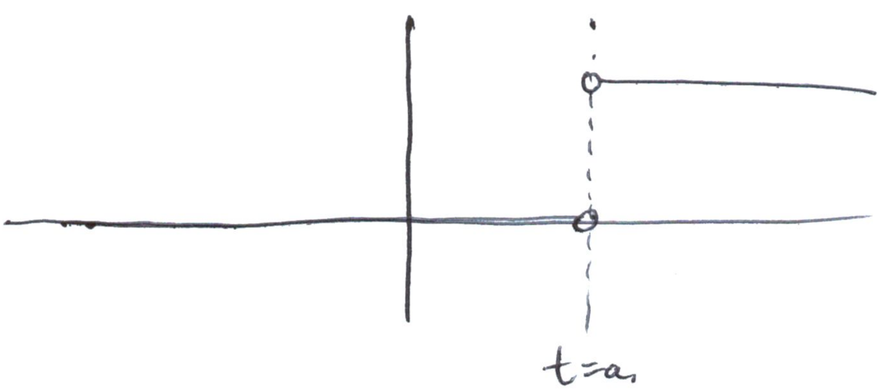
So this function looks like:



I.e. there's a jump of height 1 at $t=0$, and the function is at height 1 to the right.

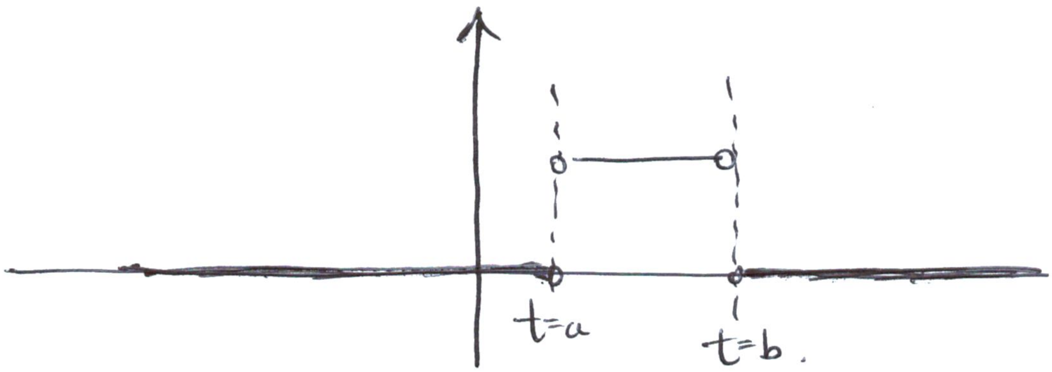
Sometimes we'll want a jump at $t=a$ instead of $t=0$. Then the function we'll want is:

$$h(t-a) = \begin{cases} 0, & t<a \\ 1, & t>a. \end{cases}$$



or we might want a function that only at height 1 over some interval [a,b]:

$$h(t-a) - h(t-b) = \begin{cases} 0 & \text{if } t < a \\ 1 & \text{if } a < t < b \\ 0 & \text{if } t > b \end{cases}$$



Then when you multiply a function $f(t)$ by a function $h(t-a)$, we want to know how the Laplace transform changes. It changes like this:

If $\mathcal{L}\{f(t)\} = F(s)$, then

$$\mathcal{L}\{f(t)h(t-a)\} = \int_0^\infty e^{-st} f(t) h(t-a)\, dt$$

$$= \int_a^\infty e^{-st} f(t)\, dt$$

set $t = u+a$.

$$= \int_0^\infty e^{-s(u+a)} f(u+a)\, du$$

$$= e^{-sa} \int_0^{\infty} e^{-su} f(u+a)\, du$$

$$= e^{-sa} \mathcal{L}\{f(t+a)\}.$$

So now we have

$$\mathcal{L}\{f(t)h(t-a)\} = e^{-sa} \mathcal{L}\{f(t+a)\}$$

How to use this? It's used for discontinuous functions!

<u>Example</u>

Find $\mathcal{L}\{f(t)\}$, where $f(t) = \begin{cases} 0 & \text{if } 0 \le t < 2 \\ t^2 - 3 & \text{if } t \geq 2. \end{cases}$

<u>Solution</u>: We can write $f(t)$ as

$$f(t) = (t^2-3)h(t-2),$$

because the step function $h(t-2)$ is <u>zero</u> to the left of $t=2$ (so there, $f(t) = (t^2-3)\cdot 0 = 0$) and to the right of $t=2$ it's 1 (so there, $f(t) = (t^2-3)\cdot 1 = t^2-3$).

Then we calculate:

$$\mathcal{L}\{(t^2-3)h(t-2)\} = e^{-2s} \mathcal{L}\{\underbrace{(t+2)^2-3}\}$$

here appears
this function shifted by 2.

$$= e^{-2s}\mathcal{L}\{t^2+4t+1\}$$

$$= e^{-2s}\left(\frac{2}{s^3}+\frac{4}{s^2}+\frac{1}{s}\right).$$

Example: Calculate the Laplace transform of

$$f(t) = \begin{cases} 0 & \text{if } t<\pi \\ \sin t & \text{if } \pi<t<2\pi \\ 0 & \text{if } t>2\pi \end{cases}$$

Solution: We can write $f(t)$ as

$$f(t) = \sin t\,(h(t-\pi) - h(t-2\pi))$$

$$= \sin(t)\,h(t-\pi) - \sin(t)\,h(t-2\pi)$$

Then

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{\sin(t)\,h(t-\pi)\} - \mathcal{L}\{\sin(t)\,h(t-2\pi)\}$$

$$= e^{-\pi s}\mathcal{L}\{\sin(t+\pi)\} - e^{-2\pi s}\mathcal{L}\{\sin(t+2\pi)\}$$

$$= e^{-\pi s}\mathcal{L}\{-\sin(t)\} - e^{-2\pi s}\mathcal{L}\{\sin(t)\}$$

$$= -e^{-\pi s}\frac{1}{s^2+1} - e^{-2\pi s}\frac{1}{s^2+1}.$$

§16.2 Continued.

Last day we saw two 'shifting formulas' given by:

$$\mathcal{L}\{e^{at} f(t)\} = F(s-a)$$

and

$$\mathcal{L}\{f(t)h(t-a)\} = e^{-sa}\mathcal{L}\{f(t+a)\}$$

where

$$h(t) = \begin{cases} 1 & \text{if } t>0 \\ 0 & \text{if } t<0 \end{cases}$$

is the unit step function.

Let's do an example of their use:

<u>Example</u>: What is the Laplace transform of

$$f(t) = \begin{cases} 0 & \text{if } 0 \leq t < 1 \\ e^{at} & \text{if } 1 < t < 2 \\ t^2 & \text{if } 2 < t \end{cases}$$

<u>Solution</u>: This function has two nonzero pieces:

<u>Piece 1</u>: The function does e^{at} from $t=1$ to $t=2$

<u>Piece 2</u>: The function does t^2 from $t=2$ onwards.

So we can write $f(t)$ as a sum of two things.

$$f(t) = e^{at}\big(h(t-1) - h(t-2)\big) + t^2 h(t-2)$$

This "turns on" e^{at} from $t=1$ to $t=2$

this "turns on" t^2 for $t>2$.

So

$$\begin{aligned}
\mathcal{L}\{f(t)\} &= \mathcal{L}\{e^{at}h(t-1)\} - \mathcal{L}\{e^{at}h(t-2)\} + \mathcal{L}\{t^2 h(t-2)\} \\
&= e^{-s}\mathcal{L}\{e^{a(t+1)}\} - e^{-2s}\mathcal{L}\{e^{a(t+2)}\} + e^{-2s}\mathcal{L}\{(t+2)^2\} \\
&= e^{-s}\mathcal{L}\{e^{a}\cdot e^{at}\} - e^{-2s}\mathcal{L}\{e^{2a}\cdot e^{at}\} + e^{-2s}\mathcal{L}\{t^2+4t+4\}. \\
&= e^{-s+a}\mathcal{L}\{e^{at}\} - e^{-2s+2a}\mathcal{L}\{e^{at}\} + e^{-2s}\mathcal{L}\{t^2+4t+4\} \\
&= e^{-s+a}\frac{1}{s-a} - e^{-2s+2a}\frac{1}{s-a} + e^{-2s}\left(\frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s}\right).
\end{aligned}$$

<u>Note</u>: We evaluated $\mathcal{L}\{e^{at}h(t-1)\}$ by doing the shift rule

$$\mathcal{L}\{f(t)\,h(t-a)\} = e^{-sa}\mathcal{L}\{f(t+a)\}$$

with $a=1$ and $f(t) = e^{at}$. However the shift rule

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a)$$

also applies, with

$$f(t) = h(t-1)$$

and $F(s) = \mathcal{L}\{h(t-1)\} = e^{-s}\mathcal{L}\{1\} = \dfrac{e^{-s}}{s}$

So we get, using this approach,

$$\mathcal{L}\{e^{at}h(t-1)\} = \frac{e^{-(s-a)}}{s-a} = \frac{e^{-s+a}}{s-a}, \text{ as before.}$$

We can also use these shifting rules when taking inverse Laplace transforms. For the purpose of taking the inverse, the formula

$$\mathcal{L}\{f(t)h(t-a)\} = e^{-sa}\mathcal{L}\{f(t+a)\}$$

can be rewritten as

$$g(t)=f(t+a)$$
$$g(t-a)=f(t)$$

$$\mathcal{L}\{f(t-a)h(t-a)\} = e^{-sa}\mathcal{L}\{f(t)\}.$$

or $\mathcal{L}\{f(t-a)h(t-a)\} = e^{-sa}F(s)$

so that $\mathcal{L}^{-1}\{e^{-sa}F(s)\} = f(t-a)h(t-a)$.

Example: Calculate the inverse Laplace transform of

$$H(s) = \frac{2e^{-3s} + 3e^{-7s}}{(s-3)(s^2+9)}.$$

Remark: This comes from Vivek Srikrishnan's notes at PSU

Solution: This breaks up into two pieces using linearity of $\mathcal{L}^{-1}$:

$$\mathcal{L}^{-1}\{H(s)\} = 2\mathcal{L}^{-1}\left\{\frac{e^{-3s}}{(s-3)(s^2+9)}\right\} + 3\mathcal{L}^{-1}\left\{\frac{e^{-7s}}{(s-3)(s^2+9)}\right\}.$$

So to apply the formula for $\mathcal{L}^{-1}\{e^{-as}F(s)\}$ to each piece, we have to first take the inverse Laplace transform of

$$F(s) = \frac{1}{(s-3)(s^2+9)}$$

in order to find $f(t)$. Partial fractions gives:

$$F(s) = \frac{A}{s-3} + \frac{Bs+C}{s^2+9}$$

solving for A, B, C we get:

$$1 = A(s^2+9) + Bs(s-3) + C(s-3)$$

$$= \cancel{As^2} + 9A + \cancel{Bs^2} - 3Bs + Cs - 3C$$

So
$$\left.\begin{aligned} A+B&=0 \\ C-3B&=0 \\ 9A-3C&=1 \end{aligned}\right\} \text{ solving, } A = \frac{1}{18},\ B = -\frac{1}{18},\ C = \frac{-3}{18}$$

So we can rewrite $F(s)$ as

$$F(s) = \frac{1}{18}\left(\underbrace{\frac{1}{s-3}}_{\text{This appears in the table}} + \underbrace{\frac{-s-3}{s^2+9}}_{\text{This needs to be broken in two to look like table entries}}\right).$$

$$F(s) = \frac{1}{18}\left(\frac{1}{s-3} + \frac{-s}{s^2+9} + \frac{-3}{s^2+9}\right)$$

So, $\mathcal{L}^{-1}\{F(s)\}$ is

$$\frac{1}{18}\left(e^{3t} - \cos(3t) - \cancel{3}\sin(3t)\right)$$

Now we use this as $f(t)$ and apply the formula

$$\mathcal{L}^{-1}\{e^{-sa}F(s)\} = f(t-a)h(t-a)$$

to each term in $\mathcal{L}^{-1}\{H(s)\}$ from before. We get:

$$\mathcal{L}^{-1}\left\{\frac{e^{-3s}}{(s-3)(s^2+9)}\right\} = h(t-3)\cdot\frac{1}{18}\left(e^{3(t-3)} - \cos(3(t-3)) - \cancel{3}\sin(3(t-3))\right)$$

$$\frac{e^{-3s}}{(s-3)(s^2+9)} = e^{-3s}F(s)$$

and

$$\mathcal{L}^{-1}\left\{\frac{e^{-7s}}{(s-3)(s^2+9)}\right\} = h(t-7)\frac{1}{18}\cdot\left(e^{3(t-7)} - \cos(3(t-7)) - \cancel{3}\sin(3(t-7))\right).$$

So our final answer is

$$\mathcal{L}^{-1}\{H(s)\} = \frac{h(t-3)}{9}\left(e^{3(t-3)} - \cos(3(t-3)) - 3\sin(3(t-3))\right)$$

$$+ \frac{h(t-7)}{6}\left(e^{3(t-7)} - \cos(3(t-7)) - 3\sin(3(t-7))\right).$$

Note that the shift formula

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a)$$

has a corresponding inverse usage as well, in the sense that a shift in the s variable will produce an e^{at} on the 't' side of things when you apply inverse Laplace. However in practice, this formula rarely comes up. ~~Remark~~

<u>Example</u>: Calculate the inverse Laplace transform of

$$H(s) = \frac{se^{-2s}}{(s-4)(s+3)}$$

<u>Solution</u>: Again, we'll use the same formula but first partial fractions on $\frac{1}{(s-4)(s+3)}$ gives

$$\frac{1}{(s-4)(s+3)} = \frac{4}{7}\left(\frac{1}{s-4}\right) + \frac{3}{7}\left(\frac{1}{s+3}\right)$$

So then the inverse Laplace of this part is

$$\frac{4}{7}e^{4t} + \frac{3}{7}e^{-3t}$$

Now the factor of e^{-2s} in our original function gives $h(t-2)$ in our answer when we apply

$$\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\,h(t-a)$$

we get $$h(t-2)\left(\frac{4}{7}e^{4(t-2)} + \frac{3}{7}e^{-3(t-2)}\right).$$