Wed, Jan 7

§10.1

A <u>sequence</u> is an infinite list of numbers

$c_1, c_2, c_3, \ldots$ etc, and we write this as $\{c_n\}_{n=1}^{\infty}$.

In some mathematically formal texts, you'll see sequences described as "functions whose domain is the set of natural numbers".

<u>What they mean</u> is:

If you have a function f like this:

$$\underbrace{\{1, 2, 3, 4, \ldots\}}_{\text{natural numbers}} \xrightarrow{f} \mathbb{R} \text{ (real numbers).}$$

Then the "outputs" of f can be listed

$$f(1) = c_1$$
$$f(2) = c_2$$
$$f(3) = c_3$$
$$\vdots$$

and they form a sequence. We'll avoid talking about functions this way, as it only complicates things and we're happy thinking of an infinite list:

$\{c_1, c_2, \ldots\}$ etc.

The elements in the list are called the <u>terms</u> of the sequence.

Example:

(1) The formula $c_n = \frac{1}{n+1}$, $n = 1, 2, 3$, etc..

defines a sequence:

$\left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\right\}$

(2) The formula $c_n = \frac{(-1)^n}{2^n}$ also defines a sequence,

recall that

$(-1)^{odd} = -1$

$(-1)^{even} = +1$,

so we get

$$\left\{\frac{-1}{2}, \frac{1}{4}, \frac{-1}{8}, \frac{1}{32}, \dots\right\}.$$

These sequences are said to be defined explicitly, because ~~#~~ of the kind of formula which defines them: If you want to know the 10^{th} term, you just plug in $n=10$.

Examples.

(1) Set $c_1 = 1$ and suppose that c_n is related to previous terms by

$$c_n = (c_{n-1})^2(-1)^n + 1$$

Then

$c_1 = 1$

$c_2 = (1)^2(-1)^2 + 1 = 2$

$c_3 = (2)^2(-1)^3 + 1 = -4 + 1 = -3$

$c_4 = (-3)^2(-1)^4 + 1 = 10.$

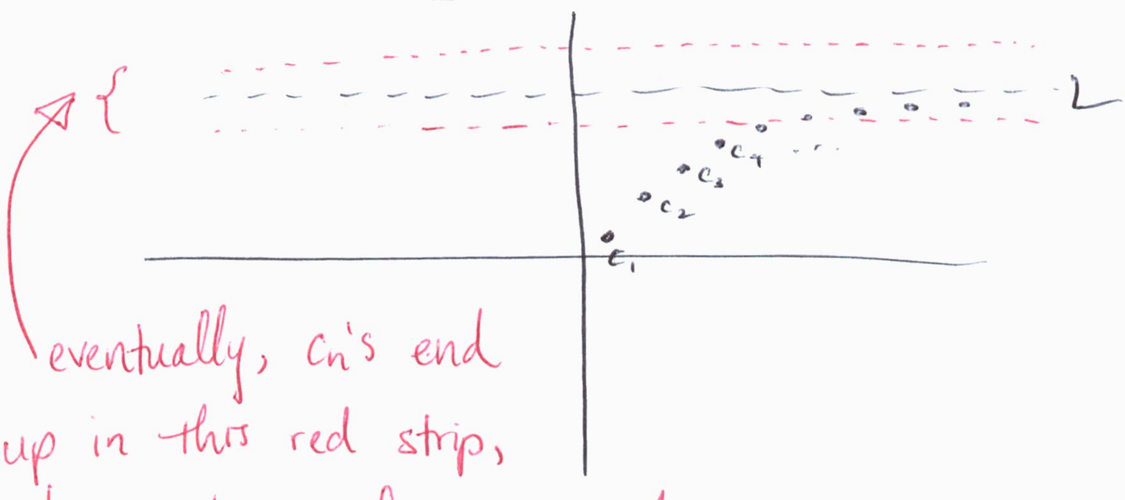
So the sequence is $\{1, 2, -3, 10\}$ and is said to be defined <u>recursively</u>, since if you want to know c_{10} you must compute $c_1, c_2, \dots, c_9$.

(2) Set $c_1 = 1$, $c_n = c_{n-1} + \sqrt{c_{n-1}^2 + 1}$.

Then

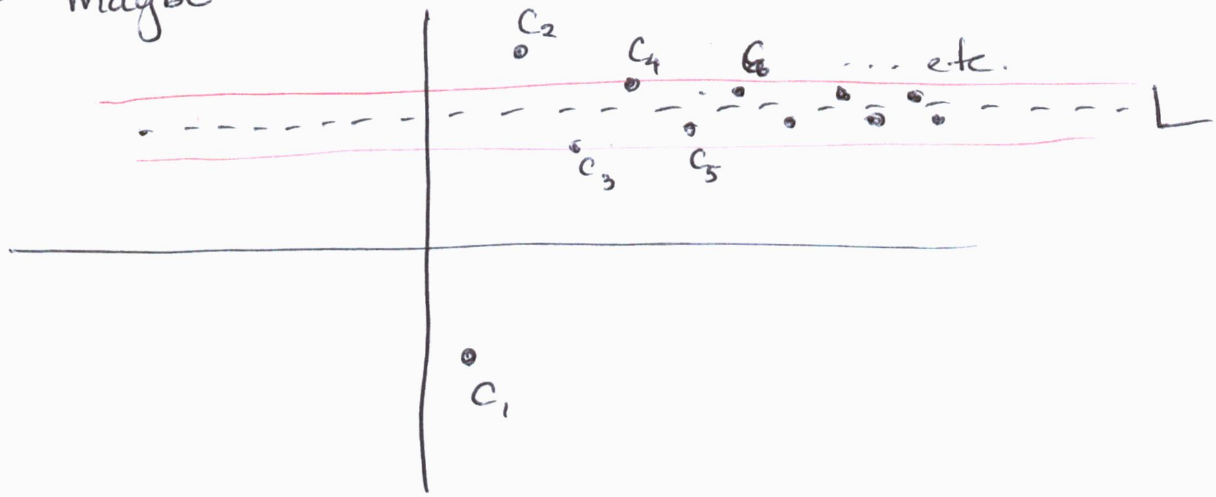
$$c_1 = 1$$
$$c_2 = 1 + \sqrt{1+1} = 1 + \sqrt{2}$$
$$c_3 = 1 + \sqrt{2} + \sqrt{(1+\sqrt{2})^2 + 1}$$
$$c_4 = \dots \text{ it gets bad.}$$

If the numbers c_n get closer and closer to some number L as n gets large, then we can plot the c_i's and they look like:



eventually, c_n's end up in this red strip, which can be as thin as we please.

or maybe



In this case, we write

$$\lim_{n \to \infty} c_n = L,$$

and say that $\{c_n\}_1^\infty$ converges. If, as we go farther and farther out in the series, c_n doesn't approach some number L

(like the sequence $c_n = n^2$)

then the sequence is said to diverge.

Example:

Oure examples

$c_n = \frac{1}{n+1}$ and $c_n = \frac{(-1)}{2^n}$ both converge to $L=0$

zero, because as $n \to \infty$, the quotient gets very small.

The recursive example

$c_1 = 1$, $c_n = (c_{n-1})^2(-1)^n + 1$ behaves in an unknown way, butt it looks like it might be diverging

~~The example $c_1 = \sqrt{2}$, $c_n = \sqrt{2 + c_{n-1}}$~~

~~converges to $L = \pi$ (incredibly).~~

The final example of

$c_1 = 1$ and $c_n = c_{n-1} + \sqrt{c_{n-1}^2 + 1}$ converges to . . .

$c_1 = 1$

$c_2 = 2.41421\ldots$

$c_3 = 5.027339\ldots$

$c_4 = 10.1531703\ldots$

$c_5 = 20.3554676$

$c_6 = 40.735\ldots$

nothing, it seems. However finding an <u>explicit</u> formula for c_n would let us check this, but finding such a formula is usually hard!

But if we do

$c_1 = 1, \quad c_{n+1} = c_n + \sin(c_n) + \frac{1}{6}\sin^3(c_n)$

it converges to π!

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Recall that a recursive sequence is defined by a formula where the n^{th} term depends on previous terms, ie.

$$c_1 = 0, \quad c_n = \frac{c_{n-1} + 10}{(c_{n-1})^2}, \quad \text{for example.}$$

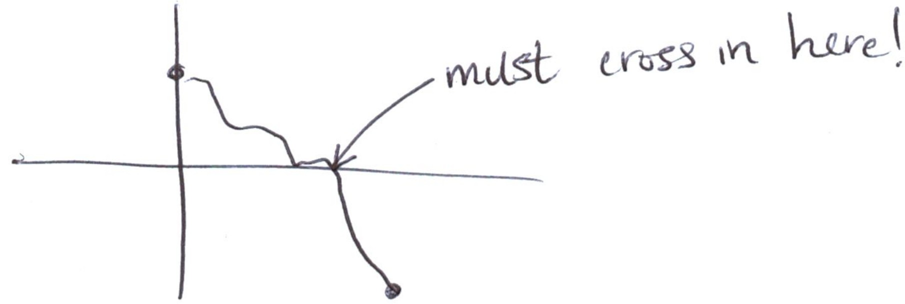
In numerical analysis of real-world problems, this is the type of sequence one often encounters when successively refining the solution to a problem.

Example: Compute the roots of $f(x) = x^3 - 3x + 1$ (ie, numerically approximate them).

First, observe that $f(0) = 0^3 - 3\cdot(0) + 1 = 1$,

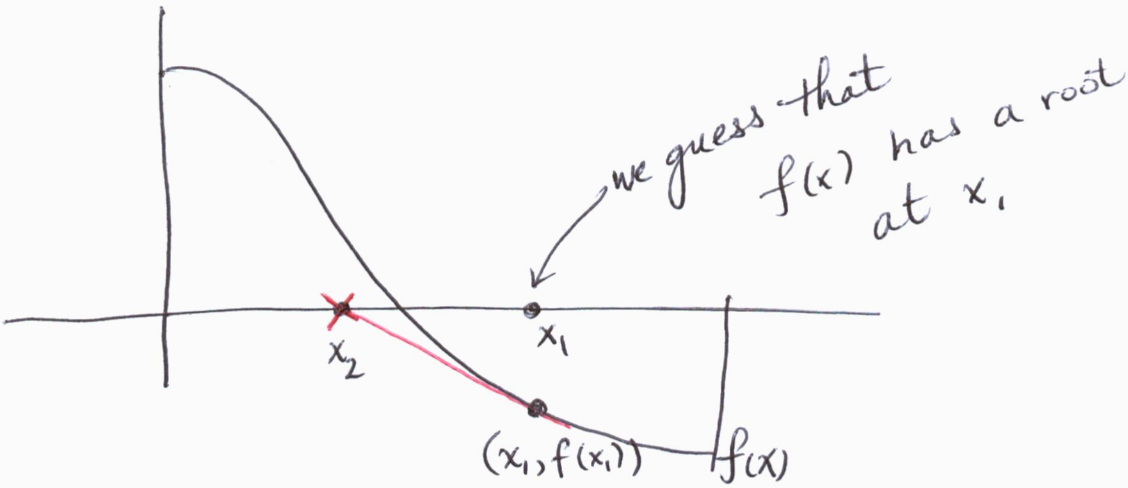
and $f(1) = 1 - 3(1) + 1 = -1$,

So the function starts above the x-axis and ends below:



So, we know there's a root in $[0, 1]$.

So we'll start with an initial guess of x_1 for the root, maybe our guess is a bit off:



We refine our guess by following the tangent line to where it intersects the x-axis. The equation of the tangent line is:

$$pt = (x_1, f(x_1)), \text{ slope} = f'(x_1)$$

so

$$y = \underbrace{f'(x_1)}_{m} \cdot x + \underbrace{f(x_1) - x_1 \cdot f'(x_1)}_{b}$$

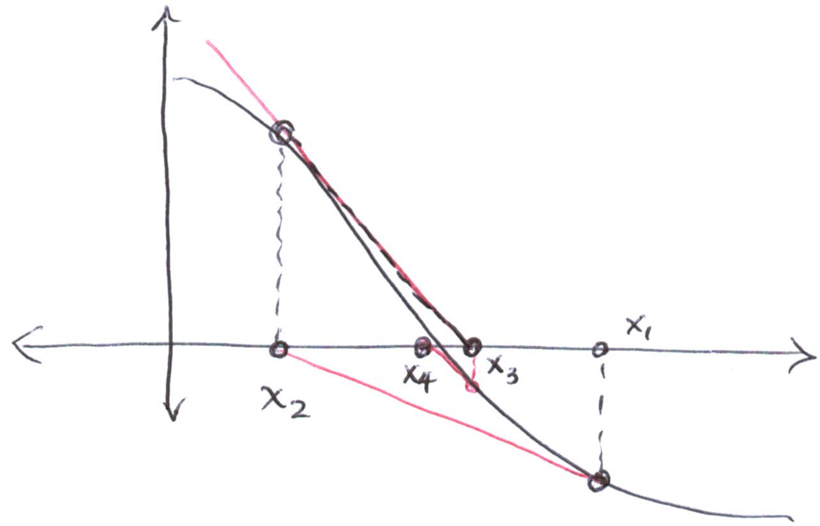
and the tangent line crosses the x-axis at

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}.$$

We can use this as our general recursive formula:

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}$$

and iterate it:



i.e. we repeatedly follow tangent lines in the direction of the root that we're wanting to approximate.

Using our original $f(x) = x^3 - 3x + 1$, and a "first guess" of $x_1 = 0.7$, we get

$$x_2 = 0.7 - \frac{f(0.7)}{f'(0.7)} = 0.205$$

$x_2 = 0.205$

$x_3 = 0.342$

$x_4 = 0.34728\ldots$

$x_5 = 0.347296\ldots$

Then if you test, we find

$f(0.3472966) = 9.374 \times 10^{-7} = 0.0000009374$,

i.e. it is <u>really</u> <u>really</u> close to zero. This approach, using recursive sequences to find the root, is

<u>Newton's</u> <u>Method</u>, or the method of ~~fix~~ <u>successive</u> <u>approximations</u>!

Example: If $f(x) = \cos(x) - x$, then

$$f(0) = \cos(0) - 0 = 1 \quad \text{and} \quad f(\tfrac{\pi}{2}) = \cos(\tfrac{\pi}{2}) - \tfrac{\pi}{2}$$
$$= \underbrace{1 - \tfrac{\pi}{2}}_{\text{negative}}$$

So $f(x)$ must cross the x-axis somewhere between 0 and $\pi/2$. We can approximate by doing:

$x_1 = 0.5$ (arbitrary guess) $\qquad f'(x) = -\sin(x) - 1$

$$x_2 = 0.5 - \frac{f(0.5)}{f'(0.5)}$$

$$= 0.75522\ldots$$

$$x_3 = 0.75522 - \frac{f(0.75522)}{f'(0.75522)}$$

$$= 0.739142\ldots$$

and if we check:

$$f(0.739142) = \cos(0.739142) - 0.739142 = -0.000095\ldots$$

so we're getting close to a root.

I.e, in the language of yesterday, this method produces a sequence $x_1, x_2, x_3, \ldots$ etc converging to a solution to $f(x) = 0$!

Next class we will study sequences of functions; and what it means for them to converge.

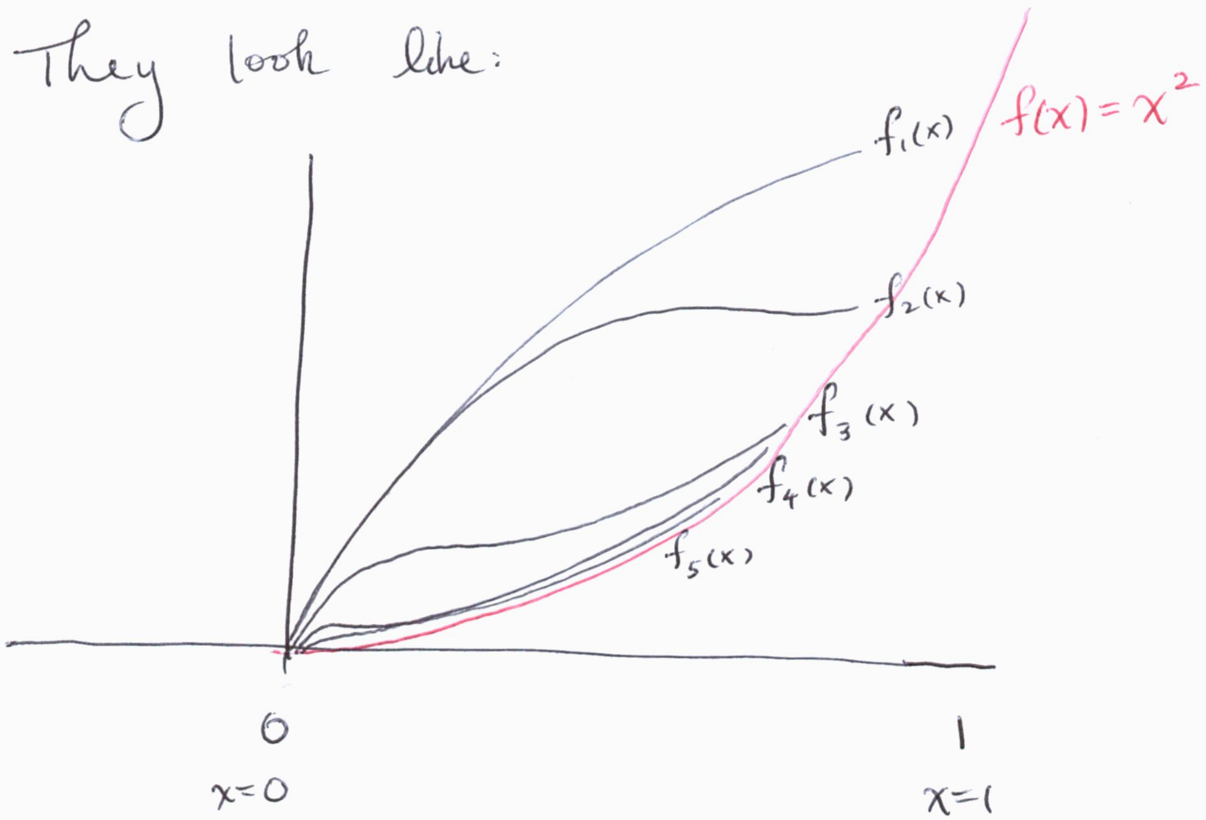
Example: Consider the functions

$$f_n(x) = x^2 + 10xe^{-nx}$$

So $f_1(x) = x^2 + 10xe^{-x}$

$f_2(x) = x^2 + 10xe^{-2x}$, ... etc.

They look like:



So as $n \to \infty$, $f_n(x)$ looks more and more like $f(x) = x^2$. This is what we mean by a converging sequence of functions. More next class.

Example: The functions $f_n(x) = \frac{|x|^n}{n!}$ provide a perfectly reasonable sequence of functions. Next day we will do a careful analysis in order to show

$$\lim_{n\to\infty} \frac{|x|^n}{n!} = 0,$$ ie the limit is the function

$f(x) = 0$ for all x.