Tutorial 9

Lots of solving IVPs!

Example: Solve $y'' + 2y' + 5y = f(t)$, $y(0) = 0$, $y'(0) = 0$

and $f(t) = \begin{cases} 4 & 0 < t < 1 \\ -4 & 1 < t < 2 \\ 0 & t > 2. \end{cases}$

Solution:

Step 1: Take $\mathcal{L}$ of both sides.

$$\mathcal{L}\{y''\} + 2\mathcal{L}\{y'\} + 5\mathcal{L}\{y\} = \mathcal{L}\{f(t)\}.$$

To take $\mathcal{L}\{f(t)\}$, we need to write $f(t)$ using step functions:

$$\begin{aligned} f(t) &= 4 \cdot (1 - h(t-1)) + (-4)(h(t-1) - h(t-2)) \\ &= 4 - 4h(t-1) - 4h(t-1) + 4h(t-2) \\ &= 4 - 8h(t-1) + 4h(t-2) \end{aligned}$$

So we get

$$\begin{aligned} s^2 Y(s) - \cancel{sy(0)} - \cancel{y'(0)} + 2(sY(s) - \cancel{y(0)}) + 5Y(s) \\ = \mathcal{L}\{4\} - 8\mathcal{L}\{h(t-1)\} + 4\mathcal{L}\{h(t-2)\} \\ = \frac{4}{s} - \frac{8e^{-s}}{s} + \frac{4e^{-2s}}{s}. \end{aligned}$$

Step 2 : Solve for $Y(s)$:

$$(s^2+2s+5)Y(s) = \frac{1}{s}(4-8e^{-s}+4e^{-2s})$$

$$\Rightarrow \quad Y(s) = \frac{1}{s(s^2+2s+5)}(4-8e^{-s}+4e^{-2s})$$

Step 3 : Calculate $\mathcal{L}^{-1}$:

We need to deal with $\frac{1}{s(s^2+2s+5)}$. First check if s^2+2s+5 factors, and find

$b^2-4ac = (2)^2-4(5)(1) = 4-20=-16$, so no real roots.

So partial fractions gives:

$$\frac{1}{s(s^2+2s+5)} = \frac{A}{s} + \frac{Bs+C}{s^2+2s+5}$$

$$\Rightarrow \quad 1 = A(s^2+2s+5)+Bs^2+Cs$$

$\Rightarrow A+B=0 \qquad B=-1/5.$

$2A+C=0 \qquad \Rightarrow C=-2/5$

$5A=1. \qquad \Rightarrow A=1/5$

So $$\frac{1}{s(s^2+2s+5)} = \frac{1}{5}\left(\frac{1}{s}-\frac{s+2}{s^2+2s+5}\right).$$

Completing the square on the second term:

$$\frac{s+2}{s^2+2s+5} = \frac{s+2}{(s+1)^2+2^2}$$

The table entries which have $(s+1)^2+2^2$ as denominator are $\dfrac{2}{(s+1)^2+2^2}$ and $\dfrac{s+1}{(s+1)^2+2^2}$, so we write

$$\frac{s+2}{(s+1)^2+2^2} = A\left(\frac{2}{(s+1)^2+2^2}\right) + B\left(\frac{s+1}{(s+1)^2+2^2}\right)$$

Equating tops:

$$s+2 = 2A + Bs + B$$

$$\Rightarrow B = 1 \text{ and } 2A+B = 2$$

$$\Rightarrow 2A = 1$$

$$\Rightarrow A = \tfrac{1}{2}.$$

Therefore

$$\frac{1}{s(s^2+2s+5)} = \frac{1}{5}\left(\frac{1}{s} - \frac{1}{2}\left(\frac{2}{(s+1)^2+2^2}\right) - \left(\frac{s+1}{(s+1)^2+2^2}\right)\right)$$

So

$$① \ \mathcal{L}^{-1}\left\{\frac{1}{s(s^2+2s+5)}\right\} = \frac{1}{5}\left(1 - \frac{1}{2}e^{-t}\sin(2t) - e^{-t}\cos(2t)\right)$$

Now incorporate the missing factors of 4, $-8e^{-s}$, and $4e^{-2s}$.

② $\mathcal{L}^{-1}\left\{-8e^{-s}\cdot\dfrac{1}{s(s^2+2s+5)}\right\}$

$$= \frac{-8}{5}h(t-1)\left[(t-1)-\frac{1}{2}e^{-(t-1)}\sin(2(t-1))-e^{-(t-1)}\cos(2(t-1))\right].$$

③ $\mathcal{L}^{-1}\left\{4e^{-2s}\cdot\dfrac{1}{s(s^2+2s+5)}\right\}$

$$= \frac{4}{5}h(t-2)\left[(t-2)-\frac{1}{2}e^{-(t-2)}\sin(2(t-2))-e^{-(t-2)}\cos(2(t-2))\right].$$

Then our answer is

$$y(t) = 4\cdot① + ② + ③$$

Example: Use convolution to solve

$$y'' + y = \tan(t), \quad y(0) = 1, \quad y'(0) = 2.$$

Solution: Here, we'll run into a problem without convolution: we can't take the Laplace transform of $\tan(t)$ using table entries! However, convolution saves us:

Step 1: Take $\mathcal{L}$ of both sides.

$$\mathcal{L}\{y''\} + \mathcal{L}\{y\} = \mathcal{L}\{\tan(t)\}$$

no idea what this is equal to, just leave it like this.

$$s^2 Y(s) - s\,y(0) - y'(0) + Y(s) = \mathcal{L}\{\tan(t)\}$$

$$\Rightarrow (s^2+1)Y(s) - s - 2 + Y(s) = \mathcal{L}\{\tan(t)\}.$$

Step 2: Solve for $Y(s)$:

$$Y(s) = \mathcal{L}\{\tan(t)\} \cdot \frac{1}{s^2+1} + \frac{2+s}{s^2+1}.$$

Step 3: Take $\mathcal{L}^{-1}$.

First, let's deal with the familiar part:

$$\mathcal{L}^{-1}\left\{\frac{+2}{s^2+1} + \frac{s}{s^2+1}\right\}$$

$$= 2\sin t + \cos t.$$

Now, the term $\mathcal{L}\{\tan(t)\}\cdot\frac{1}{s^2+1}$:

Recall that convolution works like this: If $F(s) = \mathcal{L}\{f(t)\}$ and $G(s) = \mathcal{L}\{g(t)\}$, then

$$\mathcal{L}^{-1}\{F(s)G(s)\} = f(t) * g(t) = \int_0^t f(u)\, g(t-u)\, du.$$

We can also express this as

$$\mathcal{L}^{-1}\{F(s)G(s)\} = \mathcal{L}^{-1}\{F(s)\} * \mathcal{L}^{-1}\{G(s)\}$$

So in our example:

$$\mathcal{L}^{-1}\left\{\mathcal{L}\{\tan(t)\}\cdot\frac{1}{s^2+1}\right\} = \mathcal{L}^{-1}\{\mathcal{L}\{\tan t\}\} * \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$= \tan(t) * \sin(t)$$

$$= \int_0^t \tan(u)\cdot\sin(t-u)\, du.$$

Now I'm not claiming that this integral is easy, but it IS possible, so we do it:

$$= \cos(t)\ln\left(\frac{\cos(t)}{1+\sin(t)}\right) + \sin(t).$$

So overall,

$$y(t) = 2\sin t + \cos t + \cos(t)\ln\left(\frac{\cos t}{1+\sin t}\right) + \sin(t).$$

Example: Use convolution to solve

$$y'' + 9y = \frac{1}{\sqrt{t}}, \quad y(0) = 0 \text{ and } y'(0) = 0.$$

It is Ok to leave your answer in the form of an integral

Solution:

Step 1: Take $\mathcal{L}$ of both sides

$$\mathcal{L}\{y''\} + 9\mathcal{L}\{y\} = \mathcal{L}\left\{\frac{1}{\sqrt{t}}\right\} \leftarrow \text{Leave as is.}$$

$$\Rightarrow s^2 Y(s) - sy(0) - y'(0) + 9Y(s) = \mathcal{L}\left\{\frac{1}{\sqrt{t}}\right\}$$

$$\Rightarrow (s^2+9)Y(s) = \mathcal{L}\left\{\frac{1}{\sqrt{t}}\right\}$$

Step 2: Solve for $Y(s)$.

$$Y(s) = \mathcal{L}\left\{\frac{1}{\sqrt{t}}\right\} \cdot \frac{1}{s^2+9}$$

Step 3: Take $\mathcal{L}^{-1}$!

As before, convolution comes to the rescue:

$$\mathcal{L}^{-1}\left\{\mathcal{L}\left\{\frac{1}{\sqrt{t}}\right\} \cdot \frac{1}{s^2+9}\right\} = \mathcal{L}^{-1}\left\{\mathcal{L}\left\{\frac{1}{\sqrt{t}}\right\}\right\} * \mathcal{L}^{-1}\left\{\frac{1}{s^2+9}\right\}$$

Then $\mathcal{L}^{-1}\left\{\frac{1}{s^2+9}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{3}\cdot\frac{3}{s^2+3^2}\right\} = \frac{1}{3}\sin 3t$,

so

$$\mathcal{L}^{-1}\left\{\mathcal{L}\left\{\frac{1}{\sqrt{t}}\right\}\cdot\frac{1}{s^2+9}\right\} = \frac{1}{\sqrt{t}} * \frac{1}{3}\sin 3t$$

$$= \int_0^t \frac{1}{\sqrt{u}}\cdot\frac{1}{3}\sin(3(t-u))\,du.$$

Since we are told not to evaluate the integral, our solution is

$$y(t) = \frac{1}{3}\int_0^t \frac{1}{\sqrt{u}}\sin(3(t-u))\,du.$$

Remark: If you are thinking in terms of applications, a solution of this form is actually useful, since there are many, many ways of numerically approximating an integral like this for a fixed value of t.

Convolution can even be used to solve problems for which we already know ways of doing them.

Example: Solve the initial value problem

$$y'' + 9y = f(t), \quad y(0)=1, \quad y'(0)=2 \quad \text{where}$$

$$f(t) = \begin{cases} 0 & \text{if } 0 < t < 4 \\ 1 & \text{if } t > 4. \end{cases}, \quad \text{using convolution.}$$

Solution: As always we write $f(t)$ using step functions, and we get:

$$f(t) = h(t-4).$$

Now let's do the same convolution approach as the previous two problems in order to find $y(t)$:

Step 1: Take $\mathcal{L}$ of both sides:

$$\mathcal{L}\{y''\} + 9\mathcal{L}\{y\} = \mathcal{L}\{f(t)\}$$

$$\Rightarrow s^2 Y(s) - sy(0) - y'(0) + 9Y(s) = \mathcal{L}\{f(t)\}.$$

$$\Rightarrow (s^2+9)Y(s) - s - 2 = \mathcal{L}\{f(t)\}$$

Step 2: Solve for $Y(s)$

$$Y(s) = \frac{\mathcal{L}\{f(t)\} + s + 2}{s^2+9}.$$

Step 3: Take $\mathcal{L}^{-1}$. We get:

$$y(t) = \mathcal{L}^{-1}\left\{\mathcal{L}\{f(t)\}\cdot\frac{1}{s^2+9}\right\} + \mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\} + \mathcal{L}^{-1}\left\{\frac{2}{s^2+9}\right\}$$

$$= \mathcal{L}^{-1}\{\mathcal{L}\{h(t-4)\}\} * \mathcal{L}^{-1}\left\{\frac{1}{s^2+9}\right\} + \cos 3t + \frac{2}{3}\sin 3t$$

$$= \underbrace{h(t-4) * \frac{1}{3}\sin 3t}_{} + \cos 3t + \frac{2}{3}\sin 3t.$$

so if we can evaluate the corresponding integral here, then we have an answer!

The integral is

$$\int_0^t \underbrace{h(u-4)\cdot\frac{1}{3}\sin(3(t-u))}_{}\,du$$

This function is either 0 if $h(u-4)$ is "off" or it is $\frac{1}{3}\sin(3(t-u))$ if $h(u-4)$ is "on".

So we end up with two cases:

Case 1: $0 < t < 4$ then

$$\int_0^t h(u-4)\sin(3(t-u))\,du = \int_0^t 0\,du = 0,$$

since $h(u-4)$ is off when u is between 0 and $t < 4$.

Case 2: $t > 4$. Then

$$\int_0^t h(u-4)\frac{1}{3}\sin(3(t-u))\,du$$

$$= \underbrace{\int_0^4 h(u-4)\frac{1}{3}\sin(3(t-u))\,du}_{\substack{\shortparallel \\ 0}} + \int_4^t h(u-4)\frac{1}{3}\sin(3(t-u))\,du$$

since $h(u-4)$ is "off" there

$$= \int_4^t \frac{1}{3}\sin(3(t-u))\,du =$$

$$= \left[\frac{1}{9}\cos(3(t-u))\right]_4^t = \frac{1}{9}\left(1-\cos(3(t-4))\right).$$

So our $y(t)$ is

$$y(t) = \begin{cases} \cos 3t + \frac{2}{3}\sin 3t & \text{if } 0<t<4 \\ \cos 3t + \frac{2}{3}\sin 3t + \frac{1}{9}\left(1-\cos(3(t-4))\right) & \text{if } t>4. \end{cases}$$