MATH 2132 Tutorial 8

We practice some types of "typical" exam questions.

Example: Calculate the Laplace transform of
$f(t) = e^{3t} h(t-2)$ using the definition.

Solution: "Using the definition" means we'll have to do

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = \int_0^\infty e^{-st} e^{3t} h(t-2)\,dt$$

$$= \int_2^\infty e^{-st} e^{3t}\,dt$$

$$= \lim_{b\to\infty} \int_2^b e^{(3-s)t}\,dt.$$

$$= \lim_{b\to\infty} \left[\frac{1}{3-s} e^{(3-s)t} \right]_2^b$$

$$= \lim_{b\to\infty} \left[\underbrace{\frac{1}{3-s} e^{(3-s)b}}_{\substack{\text{goes to zero} \\ \text{if } 3-s<0 \\ \Rightarrow 3<s.}} - \frac{1}{3-s} e^{(3-s)2} \right]$$

$$= \frac{-1}{3-s} e^{6-2s}.$$

Example: Calculate the Laplace transform of

$$f(t) = e^{4t}(t^2+1) + t^3 h(t-2).$$

Solution: Each piece of $f(t)$ uses a different shifting rule. The formulas are:

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a)$$

and $\mathcal{L}\{f(t)h(t-a)\} = e^{-as}\mathcal{L}\{f(t+a)\}$.

So on the first factor, e^{4t} causes a shift of -4 is the Laplace transform of t^2+1. So

$$\mathcal{L}\{t^2+1\} = \frac{2}{s^3} + \frac{1}{s}$$

and $\mathcal{L}\{e^{4t}(t^2+1)\} = \dfrac{2}{(s-4)^3} + \dfrac{1}{s-4}$

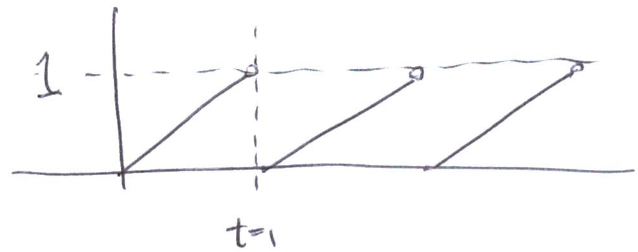
For the factor $t^3h(t-2)$, the step function introduces a factor of e^{-2s} and shifts $\mathcal{L}\{t^3\} = \frac{6}{s^4}$ by $+2$. So

$$\mathcal{L}\{t^3h(t-2)\} = e^{-2s}\cancel{\frac{6}{(s+2)^4}}.$$ Overall

$\rightarrow$ Replace with $\mathcal{L}\{(t+2)^3\}$.

$$\mathcal{L}\{f(t)\} = \frac{2}{(s-4)^4} + \frac{1}{s-4} + e^{-2s}\cancel{\frac{6}{(s+2)^4}}.$$

Example: Calculate the Laplace transform of the function $f(t) =$



where $f(t)$ has period 1.

Solution: The formula for the Laplace transform of a function $f(t)$ with period p is

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-sp}} \int_0^p e^{-st} f(t)\,dt.$$

Here, $p=1$ and $f(t)=t$ on the interval $[0,1]$. So

$$\mathcal{L}\{f\} = \frac{1}{1-e^{-s}} \int_0^1 e^{-st}\cdot t\,dt \quad (\text{by parts}) \quad \begin{matrix} u=t \\ dv=e^{-st}dt \end{matrix}$$

$$= \frac{1}{1-e^{-s}}\left(\left[t \cdot \underbrace{\frac{-e^{-st}}{s}}_{v}\right]_0^1 - \int_0^1 \frac{-1}{s} e^{-st}\,dt\right)$$

$$= \frac{1}{1-e^{-s}}\left(\frac{-e^{-s}}{s} + 0 + \frac{1}{s}\cdot\left[\left(\frac{-1}{s}\right)e^{-st}\right]_0^1\right)$$

$$= \frac{1}{1-e^{-s}}\left(\frac{-e^{-s}}{s} - \frac{1}{s^2}\left(e^{-s}-1\right)\right)$$

$$= \frac{1}{1-e^{-s}}\left(\frac{-se^{-s}-e^{-s}+1}{s^2}\right).$$

(Vivek Srikrishnan) PSU notes (not mine)

<u>Example</u>: Find the inverse Laplace transform of

$$H(s) = \frac{2}{s^3(s-1)}.$$

<u>Solution</u>: We use partial fractions, and get

$$H(s) = \frac{2}{s^3(s-1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{D}{s-1}$$

$$\Rightarrow As^2(s-1) + B(s(s-1)) + C(s-1) + Ds^3 = 2.$$

$$\Rightarrow (D+A)s^3 + (-A+B)s^2 + (C-B)s - C = 2$$

$$\Rightarrow \quad -C = 2 \Rightarrow C = -2$$

$$C - B = 0 \Rightarrow B = -2$$

$$B - A = 0 \Rightarrow A = -2$$

$$D + A = 0 \Rightarrow D = 2.$$

So we get

$$H(s) = \frac{-2}{s} - \frac{2}{s^2} - \frac{2}{s^3} + \frac{2}{s-1}$$

So then

$$\mathcal{L}^{-1}\{H(s)\} = -2 - 2t - \cancel{2}\left(\frac{t^2}{\cancel{2}}\right) + 2e^t$$

because $\mathcal{L}\{t^2\} = \frac{2}{s^3}$

$$\Rightarrow \mathcal{L}\left\{\frac{t^2}{2}\right\} = \frac{1}{s^3}$$

$$= -2 - 2t - t^2 + 2e^t.$$

Example

Find the inverse Laplace transform of

$$H(s) = \frac{4s^2 - 10s + 23}{(s^2+16)(s-1)}$$

Solution: We use partial fractions first:

$$\frac{4s^2 - 10s + 23}{(s^2+16)(s-1)} = \frac{A}{s-1} + \frac{Bs + C}{s^2+16}$$

Equate tops

$$4s^2 - 10s + 23 = As^2 + 16A + Bs^2 - Bs + Cs - C$$

$$\Rightarrow A + B = 4 \Longrightarrow B = 4 - A$$

$$C - B = -10 \Longrightarrow C - (4 - A) = -10$$

$$16A - C = 23. \qquad \Rightarrow C = -6 - A$$

So $16A - (-6 - A) = 23$

$$\Rightarrow 17A = 17$$

$A = 1$. Then $C = -7$, $B = 3$.

So

$$H(s) = \frac{1}{s-1} + \frac{3s-7}{s^2+16}.$$

So $\mathcal{L}^{-1}\{H(s)\} = e^t + \mathcal{L}^{-1}\left\{\frac{3s-7}{s^2+16}\right\}$.

to do this part

we need to make it look like table entries with denominator s^2+16, so $\frac{4}{s^2+16}$ or $\frac{s}{s^2+16}$

So we write:

$$\frac{3s-7}{s^2+16} = A\left(\frac{4}{s^2+16}\right) + B\left(\frac{s}{s^2+16}\right)$$

So we get $Bs = 3s$ and $4A = -7$

$$\Rightarrow B = 3, \quad A = \frac{-7}{4}.$$

Therefore

$$\mathcal{L}^{-1}\left\{\frac{3s-7}{s^2+16}\right\} = \frac{-7}{4}\mathcal{L}^{-1}\left\{\frac{4}{s^2+16}\right\} + 3\mathcal{L}^{-1}\left\{\frac{s}{s^2+16}\right\}$$

$$= \frac{-7}{4}\sin(4t) + 3\cos(4t)$$

So $\mathcal{L}^{-1}\{H(s)\} = e^t - \frac{7}{4}\sin(4t) + 3\cos(4t)$

Example: Calculate the inverse Laplace transform of $H(s) = \dfrac{1-3s}{s^2+2s+10}$ (vivek notes).

Solutions: First, we try to use partial fractions on it. The bottom, however, doesn't factor since

$$b^2 - 4ac = 2^2 - 4(10)(1) = 4 - 40 < 0,$$

so the quadratic formula has a negative under the square root.

So instead we complete the square on the bottom:

$$s^2 + 2s + 10$$

$$= (s+1)^2 + 9 = (s+1)^2 + 3^2.$$

So we need to take the inverse Laplace of

$\dfrac{1-3s}{(s+1)^2+3^2}$. To do this, we need to make it look like a sum of table entries. The only entries with denominator $(s+1)^2+3^2$ are $\dfrac{3}{(s+1)^2+3^2}$ and $\dfrac{s+1}{(s+1)^2+3^2}$.

So we want

$$\frac{1-3s}{(s+1)^2+3^2} = A\frac{3}{(s+1)^2+3^2} + B\frac{(s+1)}{(s+1)^2+3^2}$$

top = top gives

$$1-3s = 3A + Bs + B$$

$$\Rightarrow Bs = -3s \quad \text{and} \quad 1 = 3A + B$$

$$\Rightarrow B = -3 \quad \text{and} \quad 3A = 1 - B = 1 + 3 = 4$$

$$\Rightarrow A = \frac{4}{3}.$$

Therefore

$$\mathcal{L}^{-1}\left\{\frac{1-3s}{(s+1)^2+3^2}\right\} = \frac{4}{3}\mathcal{L}^{-1}\left\{\frac{3}{(s+1)^2+3^2}\right\} - 3\mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+3^2}\right\}.$$

$$= \frac{4}{3} e^{-t} \sin(3t) - 3 e^{-t} \cos(3t).$$

Example: Calculate the inverse Laplace transform of

$$H(s) = \frac{e^{-4s}}{s^2(s+1)}.$$

Solution: The "e^{-4s}" factor will result in a step function upon using the rule

$$\mathcal{L}^{-1}\{e^{-as} F(s)\} = f(t-a)\, h(t-a)$$

So we need to focus on the inverse of the part

$$F(s) = \frac{1}{s^2(s+1)}.$$

Partial fractions gives:

$$F(s) = \frac{1}{s^2(s+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+1}$$

$$\Rightarrow \quad 1 = As(s+1) + B(s+1) + Cs^2$$

$$\Rightarrow \quad 1 = As^2 + As + Bs + B + Cs^2$$

$$\left.\begin{aligned} \rightarrow 1 &= B \\ 0 &= A + B \\ 0 &= A + C \end{aligned}\right\} \Rightarrow \quad \begin{aligned} A &= -1 \\ C &= 1. \end{aligned}$$

So $F(s) = \frac{-1}{s} + \frac{1}{s^2} + \frac{1}{s+1}$

So

$$\mathcal{L}^{-1}\{F(s)\} = f(t) = -1 + t + e^{-t}$$

So then $\mathcal{L}^{-1}\{e^{-as}F(s)\}$ gives a step function and a shift in $f(t)$:

$$\mathcal{L}^{-1}\{H(s)\} = \mathcal{L}^{-1}\{e^{-4s}F(s)\} = h(t-4)f(t-4)$$
$$= h(t-4)\left(-1 + (t-4) + e^{-(t-4)}\right).$$

<u>Example</u>: Calculate the Laplace transform of

$$f(t) = e^{-3t}\sin(2t)\,h(t-1).$$

<u>Solution</u>: Let's <u>not</u> use the table formula for $\mathcal{L}\{e^{at}\sin(bt)\}$, and instead show how this follows from two "shifting formulas".

First, we use

$$\mathcal{L}\{f(t)\,h(t-a)\} = e^{-as}\mathcal{L}\{f(t+a)\}$$

and get

$$\mathcal{L}\{e^{-3t}\sin(2t)h(t-1)\} = e^{-s}\mathcal{L}\{e^{-3(t+1)}\sin(2(t+1))\}$$
$$= e^{-s}\mathcal{L}\{e^{-3}\cdot e^{-3t}\sin(2(t+1))\}$$
$$= e^{-s-3}\mathcal{L}\{e^{-3t}\sin(2(t+1))\}.$$

Now $\sin(2(t+1)) = \sin(2t+2)$

$= \cos 2 \sin 2t + \sin 2 \cos 2t$

So $\mathcal{L}\{\sin(2(t+1))\} = \mathcal{L}\{\cos 2 \sin 2t + \sin 2 \cos 2t\}$

$$= \frac{(\cos 2)2}{s^2+4} + \frac{(\sin 2)s}{s^2+4}.$$

Then use $\mathcal{L}\{e^{at}f(t)\} = F(s-a)$ on

$$\mathcal{L}\{e^{-3t}\sin(2(t+1))\} = \frac{(\cos 2)2}{(s+3)^2+4} + \frac{(\sin 2)s}{(s+3)^2+4}$$

So the answer is

$$\mathcal{L}\{f(t)\} = e^{-s-3}\left(\frac{(\cos 2)2}{(s+3)^2+4} + \frac{(\sin 2)s}{(s+3)^2+4}\right).$$