MATH 2132 Tutorial 7

Since last tutorial we covered the case of nonhomogeneous linear DEs with constant coefficients. In this case, the general solution is

$$y(x) = y_h(x) + y_p(x)$$

where $y_h(x)$ is a solution to the associated homogeneous problem and $y_p(x)$ is a particular solution to the original nonhomogeneous problem.

- To find y_h, use complementary eqn
- To find y_p, use undetermined coefficients.
- Add them together and apply initial conditions.

Example: (Stewart)

Solve $y'' - 4y = xe^x + \cos 2x$.

Solution: The complementary equation is

$$m^2 - 4 = 0 \Rightarrow (m-2)(m+2) = 0.$$

So the roots are ± 2. The corresponding solutions are

$$y_1(x) = e^{-2x}$$

$$y_2(x) = e^{2x}$$

and so the corresponding $y_h(x)$ is

$$y_h(x) = c_1 e^{2x} + c_2 e^{-2x}.$$

To find $y_p(x)$, we guess

$$y_p(x) = (Ax+B)e^x + C\cos 2x + D\sin 2x.$$

We can deal with the 'pieces' of y_p separately, so first we'll deal with $(Ax+B)e^x$ and solve for A, B. Call $(Ax+B)e^x$ $y_{p,1}(x)$. Then:

$$y_{p,1}'(x) = (Ax + A + B)e^x$$

$$y_{p,1}''(x) = (Ax + 2A + B)e^x$$

and so substitution gives

$$(Ax + 2A + B)e^x - 4(Ax+B)e^x = xe^x$$

Just use the part of the RHS corresponding to the piece $(Ax+B)e^x$.

$$\Rightarrow (A\cancel{x} - 4A)xe^x + (2A + B - 4B)e^x = xe^x$$

$$\Rightarrow -3A = 1, \quad 2A - 3B = 0.$$

$$\Rightarrow A = -\frac{1}{3} \text{ and } \frac{-2}{3} - 3B = 0$$

$$\Rightarrow -3B = \frac{2}{3} \Rightarrow B = \frac{-2}{9}.$$

So the first part of y_p is

$$y_{p,1}(x) = \left(-\frac{1}{3}x - \frac{2}{9}\right)e^x$$

Now with $y_{p,2}(x) = C\cos(2x) + D\sin(2x)$, we solve for C and D:

$$y'_{p,2}(x) = -2C\sin(2x) + 2D\cos(2x)$$

$$y''_{p,2}(x) = -4C\cos(2x) - 4D\sin(2x)$$

And substituting gives

$$-4C\cos(2x) - 4D\sin(2x) - 4(C\cos(2x) + D\sin(2x)) = \cos 2x.$$

$$\Rightarrow \quad -8C\cos(2x) - 8D\sin(2x) = \cos 2x$$

$$\Rightarrow -8C = 1, \quad -8D = 0$$

$$\Rightarrow C = -\frac{1}{8}, \quad D = 0.$$

So $y_{p,2}(x) = -\frac{1}{8}\cos(2x)$.

Therefore the general solution is

$$y(x) = y_h + y_{p,1} + y_{p,2}$$

$$= c_1 e^{2x} + c_2 e^{-2x} + \left(-\frac{1}{3}x - \frac{2}{9}\right)e^x - \frac{1}{8}\cos(2x).$$

<u>Example</u>: Solve $y''' + y'' = \cos(2t)$,

$$y(0) = 1, \quad y'(0) = 2, \quad y''(0) = 3.$$

<u>Solution</u>: The complementary equation is

$$m^3 + m^2 = 0 \quad \Rightarrow m^2(m+1) = 0$$

So we have $r=0$ a root of multiplicity 2,
and $r=1$ a root of multiplicity 1.
So the solutions corresponding to these roots are:

$$y_1(t) = (C_1 + C_2 t)e^{0t} = C_1 + C_2 t$$

(Note what happened here: A real root of zero 'collapses' to give a purely polynomial solution instead of exponential)

Then $y_2(t) = C_3 e^{-t}$

Therefore

$$y_h(t) = C_1 + C_2 t + C_3 e^{-t}.$$

Our guess for $y_p(t)$ is

$$y_p(t) = A\cos(2t) + B\sin(2t),$$ and we calculate:

$$y_p'(t) = -2A\overset{\sin}{\cancel{\cos}}(2t) + 2B\overset{\cos}{\cancel{\sin}}(2t)$$

$$y_p''(t) = -4A\overset{\cos}{\cancel{\sin}}(2t) - 4B\overset{\sin}{\cancel{\sin}}(2t)$$

$$y_p'''(t) = +8A\sin(2t) - 8B\cos(2t).$$

Then plugging in gives

$$8A\sin(2t) - 8B\cos(2t) - 4A\cos(2t) - 4B\sin(2t) = \cos(2t)$$

$$\Rightarrow -8B - 4A = 1 \quad \text{and} \quad 8A - 4B = 0.$$

Now solve:

$$8A - 4B = 0 \Rightarrow -4B = -8A$$

$$\Rightarrow B = 2A.$$

Then $-8(2A) - 4A = 1$

$$\Rightarrow -20A = 1 \Rightarrow A = \frac{-1}{20}, \text{ so } B = \frac{-1}{10}.$$

So the general solution is:

$$y(t) = c_1 + c_2 t + c_3 e^{-t} - \frac{1}{20}\cos(2t) - \frac{1}{10}\sin(2t).$$

Then we must use $y(0) = 1$, $y'(0) = 2$ and $y''(0) = 3$ to find c_1, c_2, and c_3:

$$y'(t) = c_2 - c_3 e^{-t} + \frac{1}{10}\sin(2t) - \frac{1}{5}\cos(2t).$$

$$y''(t) = c_3 e^{-t} + \frac{1}{5}\cos(2t) + \frac{2}{5}\sin(2t)$$

So we get

$$1 = y(0) = c_1 + c_3 - 1/20$$

$$2 = y'(0) = c_2 - c_3 - 1/5$$

$$3 = y''(0) = c_3 + 1/5.$$

$$\Rightarrow c_3 = \frac{14}{5}, \quad c_2 = 5, \quad c_1 = \frac{-7}{4}. \text{ So}$$

$$y(t) = \frac{-7}{4} + 5t + \frac{14}{5}e^{-t} - \frac{1}{20}\cos(2t) - \frac{1}{10}\sin(2t).$$

Recall that sometimes we have to make "adjustments" to our guess in order to get a correct y_p:

<u>Example</u>: Suppose that we have the differential equation
$$y^{(5)} - y^{(4)} = t^2.$$
Solve it.

<u>Solution</u>: The complementary equation is $m^5 - m^4 = 0$.

So $\quad m^4(m-1) = 0 \Rightarrow r = 0$ is multiplicity 4
$\qquad\qquad\qquad\qquad\quad r = 1$ is multiplicity 1.

So the correct y_h is
$$y_h(t) = (C_1 + C_2 t + C_3 t^2 + C_4 t^3)e^{0t} + C_5 e^t.$$
or
$$= C_1 + C_2 t + C_3 t^2 + C_4 t^3 + C_5 e^t.$$

Now our guess for $y_p(t)$ would normally be
$$y_p(t) = A + Bt + Ct^2 \quad \text{(second order polynomial)}$$
but the rule says that our guess cannot appear as part of $y_h(t)$ (which it does).

So we scale our guess by factors of t until it is <u>not</u> part of $y_h(t)$, in fact until <u>no part</u> is part of y_h!

So in this case we get

$$y_p(t) = \underbrace{At^4 + Bt^5 + Ct^6}$$

scaled so that each term <u>does not</u> appear as part of y_h.

Then we have to differentiate each:

$$y_p^{(5)}(t) = (5\cdot4\cdot3\cdot2\cdot1)B + (6\cdot5\cdot4\cdot3\cdot2)Ct$$

$$= 120B + 720Ct.$$

$$y_p^{(4)}(t) = (4\cdot3\cdot2\cdot1)A + (5\cdot4\cdot3\cdot2)Bt + (6\cdot5\cdot4\cdot3)Ct^2$$

$$= 24A + 120Bt + 360Ct^2.$$

Then plugging in:

$$(120B + 720Ct) - (24A + 120Bt + 360Ct^2) = t^2$$

$$\Rightarrow -360C = 1, \quad 720C - 120B = 0, \quad 120B - 24A = 0$$

$$\Rightarrow C = \frac{-1}{360}, \quad -2 - 120B = 0 \qquad A = \frac{120}{24}B$$

$$\Rightarrow -120B = 2 \qquad A = 5B$$

$$B = \frac{-1}{60} \qquad A = \frac{-1}{12}$$

Therefore the solution is

$$y(t) = C_1 + C_2 t + C_3 t^2 + C_4 t^3 + C_5 e^t$$
$$- \frac{1}{360} t^6 - \frac{1}{60} t^5 - \frac{1}{12} t^4.$$

(Now imagine if I gave initial values for this problem).

General facts about the midterm:

- 5 problems
  - 4 "solve this equation" type of problems, one of each kind of DE we've done so far (but no hint on each question the way to do it)
  - 1 Theory linear independence/dependence of solutions type question.
  - 10 pts per question, 70 minutes total.