Tutorial 6

§15.6. Linear operators.

Recall that an operator is <u>linear</u> if

(i) $L(x+y) = L(x) + L(y)$

(ii) $L(cx) = cL(x)$.

Are the following differential equations linear?

(17) $y'' - 3y' - 2y = 9\sec^2 x$.

<u>Solution</u>: This equation is linear, because it is written in 'standard form'

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = F(x).$$

On the other hand, we can also check linearity by verifying that the associated operator

$$\phi(D) = D^2 - 3D - 2 \quad \text{is linear.}$$

We check:

(i) $\phi(D)(y_1+y_2)$

$$= (y_1+y_2)'' - 3(y_1+y_2)' - 2(y_1+y_2)$$

$$= y_1'' + y_2'' - 3y_1' - 3y_2' - 2y_1 - 2y_2$$

$$= y_1'' - 3y_1' - 2y_1 + y_2'' - 3y_2' - 2y_2$$

$$= \phi(D)y_1 + \phi(D)y_2$$

and

(ii) $\phi(D)(cy)$

$$= (cy)'' - 3(cy)' - 2(cy)$$

$$= c[(y'') - 3y' - 2y]$$

$$= c\phi(D).$$

(19) $\sqrt{1+y'} + x^2 = 4.$

Sol If we try to isolate the terms containing y and the derivatives of y (to get it in standard linear form) then the best we can do is:

$$1 + y' = (4 - x^2)^2$$

$$\Rightarrow y' = (4 - x^2)^2 - 1$$

Now this is first order linear because it has the standard form $y' + P(x)\,y = Q(x)$, where $P(x) = 0$.

Example: Is $y'' + y^2 = x$ linear or not?

Solution: This DE is already arranged so that the left hand side contains only y's, derivatives of y and multiples/functions thereof.

However the left hand side does not behave in a linear way.

For example, suppose we plug in cy in place of y. Then we get

$$(cy)'' + (cy)^2 = x$$

$$\Rightarrow cy'' + c^2 y^2 = x$$

$$\Rightarrow c(y'' + cy^2) = x.$$

because of this c here, we cannot cleanly factor out a 'c' like $L(cy) = cL(y)$.

Similarly if we test a sum $y_1 + y_2$ plugged into the LHS:

$$(y_1+y_2)'' + (y_1+y_2)^2 = x$$

$$\Rightarrow y_1'' + y_2'' + y_1^2 + 2y_1y_2 + y_2^2 = x$$

because of this term, it's not equal to plugging in y_1 & y_2 separately:

$$(y_1'' + y_1^2) + (y_2'' + y_2^2)$$

Then we covered the following fact:

For n^{th} order linear homogeneous equations, if $y_1(x), \dots, y_n(x)$ are linearly independent then

$$c_1 y_1 + \dots + c_n y_n \text{ is a general solution.}$$

Linearly independent:

If $\quad c_1 y_1 + c_2 y_2 + \dots + c_n y_n = 0$

forces $c_1 = c_2 = c_3 = \dots = c_n = 0$ then $y_1, \dots, y_n$ are linearly independent.

Example: Show that the functions

$$y_1(x) = x, \quad y_2(x) = x^3, \quad y_3(x) = e^x$$

are linearly independent.

Solution: We write

$$c_1 y_1(x) + c_2 y_2(x) + c_3 y_3(x) = 0$$

ie. $\quad c_1 x + c_2 x^3 + c_3 e^x = 0$

and see if this forces $c_1 = c_2 = c_3 = 0$.

Plug in $x=0$:

$$c_1 \cdot 0 + c_2 \cdot 0 + c_3 e^0 = 0$$

$$\Rightarrow c_3 = 0. \checkmark$$

Now using $c_3 = 0$ oure equation is

$$c_1 x + c_2 x^3 = 0.$$

Plug in $x=1$: $c_1 + c_2 = 0 \Rightarrow c_1 = -c_2$.

Plug in $x=2$: $2c_1 + 8c_2 = 0$

$$2(-c_2) + 8c_2 = 0$$

$\Rightarrow 6c_2 = 0 \Rightarrow c_2 = 0$, so from $c_1 + c_2 = 0$

$$c_1 = 0.$$

Therefore $c_1, c_2, c_3 = 0$ and x, x^3, e^x are linearly independent.

Remark: Wronskians do this for you if you continue on in math, see q. 10 § 15.7.

Example: Find the general solution to

$$y''' - 6y'' + 12y' - 8y = 0$$

Solution:

For polynomials of degree 3 and higher, factoring is a bit of a pain. The complementary equation is:

$$m^3 - 6m^2 + 12m - 8 = 0.$$

Hint: Always test the divisors of the constant term to see if they are roots. So the divisors of 8 are: $\pm 1, \pm 2, \pm 4, \pm 8$. Test them!

We find a root of $m=2$:

$$2^3-6(2)^2+12(2)-8=8-24+24-8=0$$

So it factors as $(m-2)\cdot$ something, and we find that something using long division:

$$\begin{array}{r} m^2-4m+4 \\ m-2\,\big)\overline{m^3-6m^2+12m-8} \\ -(m^3-2m^2) \\ \hline 0-4m^2+12m \\ -(-4m^2+8m) \\ \hline 4m-8 \\ -(4m-8) \\ \hline 0. \end{array}$$

So $m^3-6m^2+12m-8=(m-2)\underbrace{(m^2-4m+4)}_{(m-2)^2}$

$$=(m-2)^3$$

So we have a root $r=2$ of multiplicity 3.

So the corresponding solution is

$$y(x)=(C_1+C_2x+C_3x^2)e^{2x}$$

and since $r=2$ is the only root, this is the general solution.

Example: Solve the following:

$$16y'' - 40y' + 25y = 0, \quad y(0) = 3 \text{ and } y'(0) = \frac{-9}{4}.$$

Solution: We can factor the complementary equation

$$16m^2 - 40m + 25 = 0$$

$$\Rightarrow (4m-5)^2 = 0.$$

and find roots $r_1, r_2 = \frac{5}{4}$

Or, you can use the quadratic equation:

$$r_1, r_2 = \frac{+40 \pm \sqrt{40^2 - 4(16)(25)}}{2(16)}$$

$$= \frac{40 \pm \sqrt{0}}{32} = \frac{5}{4}.$$

So, the solution is:

$$y(x) = (C_1 + C_2 x) e^{5/4 x}.$$

and the derivative is:

$$y'(x) = \frac{5}{4} C_1 e^{5/4 x} + C_2 e^{5/4 x} + \frac{5}{4} C_2 x e^{5/4 x}$$

Then $y(0) = 3$ gives

$$3 = (C_1 + C_2 \cdot 0) e^0 = C_1.$$

and $y'(0) = -9/4$ gives

$$-9/4 = 5/4 C_1 e^0 + C_2 e^0 + \cancel{5/4 C_2 \cdot 0 \cdot e^0}$$

$$\Rightarrow -\tfrac{9}{4} = \tfrac{5}{4} \cdot 3 + C_2$$

$$\Rightarrow \frac{-9-15}{4} = C_2 \Rightarrow C_2 = \frac{-24}{4} = -6.$$

So the solution is

$$y(x) = (3-6x)e^{5/4\,x}.$$