MATH 2132 Tutorial 6.

Recall that a first order equation is <u>separable</u> if it can be written as

$$\frac{dy}{dx} = \frac{M(x)}{N(y)}$$

and can be solved by separating and integrating:

$$\int N(y)\,dy = \int M(x)\,dx.$$

<u>Example</u>: Consider

$$\frac{dy}{dx} - x^2y^2 = x^2.$$

Solve for the solution satisfying $y(0) = 1$

<u>Solution</u>: First we solve for $\frac{dy}{dx}$ and get

$$\frac{dy}{dx} = x^2 + x^2y^2 = x^2(1+y^2).$$

Then separating:

$$\frac{1}{1+y^2}\,dy = x^2\,dx$$

and integrating:

$$\int \frac{1}{1+y^2}\,dy = \int x^2\,dx$$

We get

$$\arctan(y) = \frac{1}{3}x^3 + C.$$

Solve for y by taking tan of both sides, we get

$$y = \tan\left(\frac{1}{3}x^3 + C\right).$$

Then applying the initial condition $y(0)=1$ gives

$$1 = \tan\left(\frac{1}{3}\cdot 0^3 + C\right) = \tan(C).$$

Therefore $C = \pi/4 \pm 2k\pi$. We take $C = \pi/4$ and get $y = \tan\left(\frac{1}{3}x^3 + \frac{\pi}{4}\right)$.

Example: Find the solution to

$$\frac{dy}{dx} = 2\sqrt{y}, \quad y(0) = 4.$$

Solution: We separate

$$\frac{1}{\sqrt{y}}\,dy = 2\,dx$$

integrate

$$\int y^{-1/2}\,dy = \int 2\,dx$$

$\Rightarrow 2y^{1/2} = 2x + c$

$\Rightarrow y^{1/2} = x + C$ (here $C = \frac{\text{previous } C}{2}$, in other words we absorb the division into the constant)

$\Rightarrow y = (x + C)^2.$

With the initial condition, we then get

$4 = y(0) = (0 + C)^2$

$\Rightarrow C^2 = 4 \Rightarrow C = \pm 2.$

So there are two candidate solutions:

$y = (x+2)^2$ and $y = (x-2)^2.$

For these solutions to work out, they need to satisfy the given differential equation.

We check:

$$\frac{d}{dx}(x \pm 2)^2 \stackrel{?}{=} 2\sqrt{(x \pm 2)^2}$$

$$\Rightarrow 2(x \pm 2) \stackrel{?}{=} 2\sqrt{(x \pm 2)^2}$$

but recall that $\sqrt{z^2} = |z|$, so this means

$$2(x \pm 2) = 2|x \pm 2|.$$

Depending on the x value we plug in, this equation may or may not be true.

At least it must be true at the value $x=0$, because that is the x-value of the initial condition $y(0)=4$.

So with $x=0$ we get two possibilities:

$$2(0+2) = 2|0+2|$$
$\Rightarrow 4 = 4$, which is true, or

$$2(0-2) = 2|0-2|$$
$\Rightarrow -4 = 4$, which is <u>not</u> true.

So the correct solution is $y = (x+2)^2$.

<u>Example</u>: Sometimes you cannot solve for y, then it is ok to give an <u>implicit solution</u>.

For example,

$$\frac{dy}{dx} = \frac{x+1}{8+2\pi \sin(\pi y)}$$

is separable and has solution y satisfying:

$$8y - 2\cos(\pi y) = \frac{1}{2}x^2 + x + C.$$

Here, you cannot solve for y so it is left in this form.

Recall an equation is first-order linear if it can be written as

$$\frac{dy}{dx} + P(x)y = Q(x).$$

Then the general solution is

$$y = \frac{\int Q(x) e^{\int P(x)dx} dx + C}{e^{\int P(x)dx}}$$

where

$e^{\int P(x)dx} = \mu(x)$ is called the <u>integrating factor</u>.

<u>Example</u>: Solve the DE

$$\frac{dy}{dx} + 3x^2 y = 6x^2$$

<u>Solution</u>:

We have $P(x) = 3x^2$, $Q(x) = 6x^2$, so

$$\mu(x) = e^{\int P(x)dx} = e^{x^3}$$

and then

$$\int Q(x)\mu(x)dx = \underbrace{\int e^{x^3} \cdot 6x^2 dx}_{\text{substitute to integrate } u = x^3}$$

$$= \int 2e^u du$$

$$= 2e^{x^3} + C$$

So the solution is

$$y = \frac{2e^{x^3}+C}{e^{x^3}} = 2 + e^{-x^3}C.$$

Example: Find the solution to

$$x^2y' + xy = 1, \quad x>0 \quad y(1) = 2.$$

Solution:

We put the equation into the correct form:

$$y' + \frac{1}{x}y = \frac{1}{x^2}$$

and identify $P(x) = 1/x$, $Q(x) = 1/x^2$.

Then $\mu(x) = e^{\int \frac{1}{x}dx} = e^{\ln|x|} = |x| = x$ since $x>0$.

Then $\int Q(x)\mu(x)dx = \int \frac{1}{x^2} \cdot x\, dx = \int \frac{1}{x}dx = \ln(x)+C$

So the general solution is

$$y = \frac{\ln(x) + C}{x}.$$

Then $y(1) = 2$, so

$$2 = \frac{\ln(1) + C}{1} = \frac{0 + C}{1} \Rightarrow C = 2.$$

So the solution is $y = \dfrac{\ln(x)+2}{x}$.

We also saw 2 kinds of second-order and one kind of first-order equation that can be reduced to separable/linear by substitution:

The kinds are:

(i) A second order DE containing no y.
Then set $v = y'$, $v' = y''$, solve.

(ii) A second order DE containing no x.
Then set $v = \frac{dy}{dx}$ and $\frac{d^2y}{dx^2} = v\frac{dv}{dy}$.

(iii) Bernoulli equations

$$\frac{dy}{dx} + P(x)y = Q(x)y^n.$$

Example: Solve $y' + \frac{4}{x}y = x^3y^2$, $y(2) = -1$ $x > 0$.

Solution: This is Bernoulli with $n=2$, so set $v = y'$.

Now divide by y^2:

$$\frac{1}{y^2}\frac{dy}{dx} + \frac{4}{xy} = x^3$$

Then substituting gives $v = y^{-1}$, $v' = -y^{-2}y'$
(implicit diff)

so we get

$$-v' + \frac{4}{x}v = -x^3.$$

Now we solve for v:

$$v' - \frac{4}{x}v = -x^3, \quad \text{so } P(x) = \frac{-4}{x}, \; Q(x) = -x^3$$

we calculate $\mu(x) = e^{\int P(x)\,dx} = e^{-4\ln|x|}$

$$= e^{-\ln|x^4|} = x^{-4}$$

Then $\int Q(x)\mu(x)\,dx = \int -x^3 \cdot x^{-4}\,dx = -\ln|x| + C$

$$= -\ln(x) + C$$

since $x > 0$.

So $v(x) = \dfrac{-\ln(x) + C}{x^{-4}} = x^4(C - \ln(x))$

or in other words, $y^{-1} = x^4(C - \ln(x))$

$$\Rightarrow y = \frac{1}{x^4(C - \ln(x))}.$$

Then $y(2) = -1$ gives

$$-1 = \frac{1}{16(C - \ln(2))} \Rightarrow C = \ln 2 - \frac{1}{16}.$$

So $$y = \frac{1}{x^4(\ln 2 - \frac{1}{16} - \ln(x))}.$$

Example: Solve $1 + 2y^2 \frac{d^2y}{dx^2} = 0$, with $y(0)=1$, $y'(0)=1$.

Solution: There is not any 'x' in this eqn, so set $v = \frac{dy}{dx}$ and $\frac{d^2y}{dx^2} = v\frac{dv}{dy}$.

We get
$$1 + 2y^2 v \frac{dv}{dy} = 0$$
or $$v\frac{dv}{dy} = \frac{-1}{2y^2}$$
$$\Rightarrow \int v\, dv = -\int \frac{1}{2y^2}\, dy$$
$$\Rightarrow \frac{v^2}{2} = \frac{1}{2y} + C$$

Now to save time, we can use $y'(0)=1$ here and get

$1 = y'(0)$ and $1 = y(0)$ $\Longrightarrow \frac{1^2}{2} = \frac{1}{2(1)} + C \Rightarrow C = 0.$

So $$\frac{v^2}{2} = \frac{1}{2y} \quad \text{or} \quad \frac{1}{2}\left(\frac{dy}{dx}\right)^2 = \frac{1}{2y}$$

So $\frac{dy}{dx} = \pm \frac{1}{\sqrt{y}}$ ~~or~~. But $y(0)=1$ and $y'(0)=1$ gives

$$1 = \pm \frac{1}{\sqrt{1}} = \pm 1,$$

so we should ~~by~~ keep the '+' above and not the '−'. So

$$\frac{dy}{dx} = + \frac{1}{\sqrt{y}} \text{ or}$$

$$\int \sqrt{y}\, dy = \int dx$$

$$\Rightarrow \frac{2}{3} y^{3/2} = x + C$$

But $y(0)=1$ gives $\frac{2}{3}(1)^{3/2} = 0 + C$

$$\Rightarrow C = \frac{2}{3}.$$

So $$\frac{2}{3} y^{3/2} = x + \frac{2}{3}$$

or

$$y = \left(1 + \frac{3}{2}x\right)^{2/3}.$$