MATH 2132 Tutorial

Question 2, 2009 Test 1

Let $f(x) = \dfrac{4x}{1-4x}$ for $-\frac{1}{4} < x \le \frac{1}{8}$. You are given that

$$f^{(n)}(x) = \frac{4^n n!}{(1-4x)^{n+1}} \quad \text{where } n \ge 1.$$

(a) Find the first 3 terms in the MacLaurin series for $f(x)$.

Solution: The MacLaurin series has formula

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$$

So here, $f(0) = 0$

$$f'(0) = \frac{4^1 \cdot 1!}{(1-0)^{n+1}} = 4$$

bottom is always 1 so only compute top

$$f''(0) = 4^2 \cdot 2! = 16 \cdot 2 = 32$$

$$f'''(0) = 4^3 \cdot 3! = 64 \cdot 6 = 384$$

So first 3 terms are $0 + 4x + \dfrac{32}{2!}x^2 + \dfrac{384}{3!}x^3$

$$= 4x + 16x^2 + 64x^3$$

b) Find the n^{th} remainder (the book calls it $R_n(0,x)$)

Solution: The formula is

$$R_n = \frac{f^{(n+1)}(z_n)}{(n+1)!}(x-c)^{n+1}$$ where z_n is between x and c, here $c=0$

$$= \frac{4^{n+1}\cancel{(n+1)!}}{(1-4z_n)^{n+2}} \cdot \frac{x^{n+1}}{\cancel{(n+1)!}}$$

$$= \frac{(4x)^{n+1}}{(1-4z_n)^{n+2}}$$

c) Show that $\lim_{n\to\infty} R_n(0,x)=0$ when $x<0$.

Solution: We want to take the limit:

$$\lim_{n\to\infty} \frac{4^{n+1}x^{n+1}}{(1-4z_n)^{n+2}}$$ where $x<0$ and z_n is between x and 0.

Note: If we take arbitrary $x<0$, this doesn't work! Say $x=-2$ and $z_n=-1$.

Then we'd have

$$\lim_{n\to\infty} \frac{(-8)^{n+1}}{(1+4)^{n+2}} = \lim_{n\to\infty} \left(\frac{-8}{5}\right)^{n+1} \cdot \frac{1}{5},$$ which

does not go to zero!

So it is important to note that in part (a), they say $-\frac{1}{4} < x \leq \frac{1}{8}$.

So $x<0$ means $-\frac{1}{4} < x < z_n < 0$. From this, we get $z_n - \frac{1}{4} < x$ and so $4z_n - 1 < 4x$.

Then multiply by $\dfrac{1}{4z_n - 1}$ and get:

$$1 > \frac{4x}{4z_n - 1} > 0 \quad \text{(signs change direction since } 4z_n - 1 < 0\text{).}$$

So then

$$\lim_{n\to\infty} \frac{(4x)^{n+1}}{(1-4z_n)^{n+2}} = \lim_{n\to\infty} \underbrace{\left(\frac{4x}{\cancel{4}\,1-4z_n}\right)^{n+1}}_{\substack{\text{goes to}\\ \text{zero since it is}\\ \lim_{n\to\infty} r^n \text{ where } |r|<1.}} \cdot \frac{1}{1-4z_n}$$

4. Find the sum of the series:

$$-\frac{\sqrt{2}}{3}x^3 + \frac{2}{9}x^6 - \frac{2\sqrt{2}}{27}x^9 + \dots + \frac{(-1)^n 2^{n/2}}{3^n}x^{3n}.$$

Solution: We try to recognize the sum

$$\sum_{n=1}^{\infty} \frac{(-1)^n 2^{n/2}}{3^n} x^{3n}.$$

Set $y = x^3$. Then we get

$$\sum_{n=1}^{\infty} \frac{(-1)^n 2^{n/2}}{3^n} y^n, \text{ does this help?}$$

Group all terms with like powers. Note $2^{n/2} = (\sqrt{2})^n$.

$$= \sum_{n=1}^{\infty} \left(-\frac{\sqrt{2}\,y}{3}\right)^n .$$

So this is almost $\sum_{n=0}^{\infty} a x^n$ with $a=1$, x replaced by $-\frac{\sqrt{2}\,y}{3}$.

But it starts at 1, so:

$$= \sum_{n=0}^{\infty} \left(\frac{-\sqrt{2}\,y}{3}\right)^n - 1$$

$$= \frac{1}{1+\frac{\sqrt{2}}{3}y} - 1 = \frac{1}{1+\frac{\sqrt{2}}{3}x^3} - 1.$$ This holds for

$$-1 < \frac{\sqrt{2}}{3}x^3 < 1 \iff -\frac{3}{\sqrt{2}} < x^3 < \frac{3}{\sqrt{2}}$$

$$\iff +\sqrt[3]{\frac{-3}{\sqrt{2}}} < x < \sqrt[3]{\frac{3}{\sqrt{2}}}.$$

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Find the Taylor series for $f(x) = \sqrt{x+3}$ about $x=2$.

Solution: Set $y = x-2$, then find the MacLaurin series in y. We get

$$f(x) = \sqrt{y+2+3} = (y+5)^{\frac{1}{2}}. \cancel{= \left(\frac{1}{5}\right)^{\frac{1}{2}}\left(1+5y\right)^{\frac{1}{2}}}$$

This is binomial formula.

$$= 5^{\frac{1}{2}}\left(1+\frac{y}{5}\right)^{\frac{1}{2}}.$$

Recall

$$(1+x)^m = 1 + \sum_{n=1}^{\infty} \frac{m(m-1)(m-2)\cdots(m-n+1)}{n!} x^n.$$

So our case becomes

$$f(x) = \sqrt{5}\left(1+\frac{y}{5}\right)^{\frac{1}{2}} = \sqrt{5}\left(1 + \sum_{n=1}^{\infty} \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\cdots\left(\frac{1}{2}-n+1\right)}{n!}\left(\frac{y}{5}\right)^n\right).$$

$$= \sqrt{5}\left(1 + \sum_{n=1}^{\infty} \frac{\frac{1}{2}\cdot\frac{-1}{2}\cdot\frac{-3}{2}\cdot\frac{-5}{2}\cdots\frac{3-2n}{2}}{n!}\left(\frac{y}{5}\right)^n\right)$$

$$= \sqrt{5} + \sqrt{5}\sum_{n=1}^{\infty} (-1)^{n-1}\frac{3\cdot5\cdot7\cdot\ldots\cdot 2n-3}{n!\,2^n\cdot5^n} y^n \qquad -1<y<1$$

$$= \sqrt{5} + \sqrt{5}\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\;3\cdot5\cdot7\cdots\cdot 2n-3}{n!\,10^n}(x-2)^n$$

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Find the sum of $\sum_{n=1}^{\infty} (n^2+2n)x^n$.

Solution: As a first step, we break it into two sums we can consider separately or factor $n^2+2n = n(n+2)$ and integrate. Let's try both:

Method 1:

$$\sum_{n=1}^{\infty}(n^2+2n)x^n = \sum_{n=1}^{\infty} n^2x^n + 2\sum_{n=1}^{\infty} n x^n.$$

$$= x\sum_{n=1}^{\infty} n^2x^{n-1} + 2\sum_{n=1}^{\infty} n x^{n}.$$

Now it's set up so that when you integrate either one, you get cancelation. Yesterday we did $\sum_{n=1}^{\infty} n^2x^{n-1}$ in class, and found

$$\sum_{n=1}^{\infty} n^2x^{n-1} = \frac{x+1}{(1-x)^3} \quad \text{(it took some work)}$$

$$-1 < x < 1$$

Now we can integrate

$$\int \sum_{n=1}^{\infty} n^2x^{n-1} = \sum_{n=1}^{\infty} nx^n = \underbrace{\int \frac{x+1}{(1-x)^3}\,dx}_{\text{not straightforward, but doable.}} = \frac{x}{(1-x)^2}$$

So then

$$\sum_{n=1}^{\infty}(n^2+2n)x^n = x\sum_{n=1}^{\infty} n^2x^{n-1} + 2\sum_{n=1}^{\infty} n x^n$$

$$= \frac{x(x+1)}{(1-x)^3} + \frac{2x}{(1-x)^2}$$

$$= \frac{x^2+x+2x-2x^2}{(1-x)^3} = \frac{x(3-x)}{(1-x)^3}$$

Method 2:

$$s(x) = \sum_{n=1}^{\infty} (n^2+2n)x^n = \sum_{n=1}^{\infty} n(n+2)x^n$$

So $$\int s(x) = \sum_{n=1}^{\infty} \frac{n(n+2)}{n+1} x^{n+1} + C$$

$$\Rightarrow \iint s(x) = \sum_{n=1}^{\infty} \frac{n}{n+1} x^{n+2} + Cx + D.$$

Try to make this look like $\ln(x)$. Replace $m=n+1$.

Then $$\iint s(x) = \sum_{m=2}^{\infty} \frac{m-1}{m} x^{m+1}$$

$$= \sum_{m=2}^{\infty} \left(\frac{m}{m} - \frac{1}{m}\right) x^{m+1}$$

$$= \sum_{m=2}^{\infty} x^{m+1} - \sum_{m=2}^{\infty} \frac{x^{m+1}}{m}$$

These look familiar.

$$= x^3 \sum_{n=0}^{\infty} x^n - x \sum_{m=2}^{\infty} \frac{x^m}{m}$$

$$= \frac{x^3}{1-x} - x\left(\sum_{m=1}^{\infty} \frac{x^m}{m} - x\right)$$

$$= \frac{x^3}{1-x} - x\left(-\ln(1-x) - x\right) \qquad \left(\ln(1-x) = \sum_{m=1}^{\infty} -\frac{x^m}{m} \quad -1<x<1.\right)$$

So finally

$$\iint s(x) = \frac{x^3}{1-x} + x\ln(1-x) + x^2 + Cx + D.$$

$-1 < x < 1.$

To get the answer, we differentiate twice. This gives

$$s(x) = \frac{(x-3)x}{(x-1)^3}.$$

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Find the sum of $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2n-1)}{(2n)!} x^{2n+1}$

Solution: This is a bit ridiculous, because

$$2n! = 1 \cdot 2 \cdot \ \ldots \ \cdot (2n-2) \cdot (2n-1) \cdot 2n$$

So we could rewrite it as $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\cancel{(2n-1)}}{1 \cdots (2n-2)\cancel{(2n-1)}2n} x^{2n+1}$

Now observe that we almost have $(2n)!$ on the bottom, which makes me think of cosine.

To get $(2n-1)$ to appear on the bottom

We could also split the sum as before:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2n-1)}{(2n)!} x^{2n+1} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\, 2n}{(2n)!} x^{2n+1} - \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n)!} x^{2n+1}$$

Dealing with each sum;

$$- \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n)!} x^{2n+1} = x \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$= x\left(\sum_{n=0}^{\infty} \left(\frac{(-1)^n x^{2n}}{(2n)!}\right) - 1\right)$$

$$= x\cos(x) - x. \qquad R = \infty.$$

The first sum.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)!} x^{2n+1} \qquad \text{set } 2m = 2n-2, \text{ so } \quad 2n-1 = 2m+1$$

$$2n+1 = 2m+3.$$

$$\sum_{m=0}^{\infty} \frac{(-1)^{m+2}}{(2m+1)!} x^{2m+3} \qquad 2n+2 = 2m+4$$

$$\Rightarrow n+1 = m+2$$

$$= +x^2 \sum_{m=0}^{\infty} \frac{(-1)^{m\cancel{+2}}}{(2m+1)!} x^{2m+1} = x^2 \sin x$$

so overall, our sum is

$$x^2 \sin x + x\cos x - x, \quad \text{for all } x$$

(ie $R=\infty$).