MATH 2132 Tutorial 3

Recall that we have a couple tricks for manipulating power series to find the sum:

(i) Take an infinite sum and rearrange it to look familiar

(ii) Use termwise differentiation or integration to make it look familiar.

Then once we know the limit, there are a couple formulas to figure out where it holds:

<u>Theorem</u>: If we have $f(x) = \sum_{k=0}^{\infty} a_k (x-c)^k$, then the formula is true (ie. the series converges) for $c - R < x < c + R$ where

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| \quad \text{or} \quad R = \lim_{n\to\infty} \frac{1}{\sqrt[n]{|a_n|}}$$

if either limit exists or is infinity.

<u>Examples now</u>:

<u>Example</u>: Calculate the radius of convergence of

$$\sum_{n=0}^{\infty} \frac{2^n}{n3^{n+1}} x^n$$

<u>Solution</u>: We'll try $\lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$ first. We get:

$$\lim_{n\to\infty}\left|\frac{\frac{2^n}{n\,3^{n+1}}}{\frac{2^{n+1}}{(n+1)3^{n+2}}}\right| = \lim_{n\to\infty}\left|\frac{(n+1)\,3^{n+2}\,2^n}{2^{n+1}\,n\,3^{n+1}}\right|$$

$$= \lim_{n\to\infty}\frac{3}{2}\frac{(n+1)}{n} = \frac{3}{2}.$$

So the radius of convergence is $R=\frac{3}{2}$.

Example:

Where does the ~~sequence~~ series

$$\sum_{n=1}^{\infty}\frac{1\cdot3\cdot5\cdot\ldots\cdot(2n+1)}{2\cdot5\cdot8\cdot11\cdot\ldots\cdot(3n+2)}(2x)^n \text{ converge?}$$

Solution: Let's try the limit of $\frac{a_n}{a_{n+1}}$ again:

Here, $a_n = 2^n\frac{1\cdot3\cdot5\cdot\ldots\cdot(2n+1)}{\underbrace{2\cdot5\cdot8\cdot\ldots\cdot(3n+2)}_{\text{incrementing by 3 each time.}}}$

So,

$$R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{\frac{2^n\cdot1\cdot3\cdot5\cdot\ldots\cdot(2n+1)}{2\cdot5\cdot8\cdot\ldots\cdot(3n+2)}}{\frac{2^{n+1}\cdot1\cdot3\cdot5\cdot\ldots\cdot(2n+3)}{2\cdot5\cdot8\cdot\ldots\cdot(3n+5)}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{2^n\cdot1\cdot3\cdot5\cdot\ldots\cdot(2n+1)\cdot2\cdot5\cdot8\cdot\ldots\cdot(3n+3)(3n+5)}{2^{n+1}\cdot1\cdot3\cdot5\cdot\ldots\cdot(2n+1)(2n+3)\cdot2\cdot5\cdot8\cdot\ldots\cdot(3n+3)}\right|$$

$$= \lim_{n\to\infty}\left|\frac{3n+5}{2(2n+3)}\right|$$

$$= \lim_{n\to\infty}\left|\frac{3n+5}{4n+6}\right| = \frac{3}{4}.$$

So the series converges for $-\frac{3}{4} < x < \frac{3}{4}$.

<u>Example</u>: Find the interval of convergence of

$$\sum_{n=1}^{\infty} \frac{1}{n^2 \ln(n)} x^n.$$

<u>Solution</u>: Again, let's try the ratio test to find R:

$$R = \lim_{n\to\infty} \left| \frac{\frac{1}{n^2 \ln n}}{\frac{1}{(n+1)^2 \ln(n+1)}} \right| = \lim_{n\to\infty} \left| \frac{(n+1)^2 \ln(n+1)}{n^2 \ln n.} \right| .$$

now as $n\to\infty$ we get $\frac{\infty}{\infty}$.

so we can apply L'Hôpital's rule to the limit.

$$\lim_{x\to\infty} \frac{(x+1)^2 \ln(x+1)}{x^2 \ln(x)} \qquad x > 1 \quad \text{to get the answer.}$$

We calculate: $\frac{d}{dx}(x+1)^2 \ln(x+1) = 2(x+1)\ln(x+1) + (x+1)^2 \cdot \frac{1}{x+1}$

$$= (x+1)(2\ln(x+1)+1).$$

and $\frac{d}{dx}(x^2)\ln(x) = x(2\ln(x)+1)$.

$$= \lim_{x\to\infty} \frac{2(x+1)\ln(x+1) + (x+1)}{x(2\ln(x)+1)}.$$

L' Hopital again,

$\frac{d}{dx}(\text{top}) = 2\ln(x+1)+3$

$$= \lim_{x\to\infty} \frac{2\ln(x+1)+3}{2\ln(x)+3}$$

$\frac{d}{dx}(\text{bottom}) = 2\ln(x)+3.$

This limit will definitely go to 1 as $x \to \infty$, but if you want to be 100% certain apply L'Hôpital again:

$$= \lim_{x \to \infty} \frac{\frac{2}{x+1}}{\frac{2}{x}}$$

$$= \lim_{x \to \infty} \frac{x}{x+1} = 1.$$

So the series converges for $-1 < x < 1$.

Example: Find the Maclaurin series for

$$f(x) = \frac{x^2}{3-4x}.$$

Solution: Here, $f(x) = x^2 \left(\frac{1}{3-4x} \right)$

$$= x^2 \cdot \frac{1}{3} \left(\frac{1}{1 - \frac{4}{3}x} \right)$$

$$= x^2 \left(\frac{1/3}{1 - 4/3\, x} \right)$$

$$= x^2 \sum_{n=0}^{\infty} \frac{1}{3} \left(\frac{4}{3}x \right)^n = x^2 \sum_{n=0}^{\infty} \frac{4^n}{3^{n+1}} x^n$$

$$= \sum_{n=0}^{\infty} \frac{4^n}{3^{n+1}} x^{n+2}.$$

for $-1 < \frac{4}{3}x < 1$

$$\Rightarrow -3/4 < x < 3/4.$$

Example: Now find the Taylor series for

$$f(x) = \frac{x^2}{3-4x} \quad \text{centered at} \quad x = 2.$$

Solution: If we take $y = x-2$, then what we're asking for is the Taylor series of

$$f(x) = \frac{(y+2)^2}{3-4(y+2)} = \frac{(y+2)^2}{-5-4y} \quad \text{centered at } y=0.$$

We can do this with our typical formulas:

$$f(x) = (y+2)^2\left(\frac{1}{-5-4y}\right)$$

$$= (y+2)^2\left(\frac{+1}{-5}\right)\left(\frac{1}{1-\left(\frac{-4}{5}y\right)}\right)$$

$$= (y+2)^2 \sum_{n=0}^{\infty}\left(\frac{-1}{5}\right)\left(\frac{-4}{5}y\right)^n \quad \text{for } -1 < \frac{-4}{5}y < 1.$$

ie. $-1 < \frac{-4}{5}(x-2) < 1$.

$$= (y+2)^2 \sum_{n=0}^{\infty} \frac{(-1)^{n+1}4^n}{5^{n+1}} y^n$$

Now change to x:

$$= x^2 \sum_{n=0}^{\infty} \frac{(-1)^{n+1}4^n}{5^{n+1}}(x-2)^n$$

Now the challenging part: How to make x^2 into a sum of powers of $(x-2)$?

Observe $x^2 = (x-2)^2 + \underbrace{4x-4}_{\text{add this to correct}}$

$$= (x-2)^2 + 4(x-2) + \underbrace{4}_{\text{add to correct.}}$$

So the Taylor series is:

$$\left((x-2)^2 + 4(x-2) + 4\right) \sum_{n=0}^{\infty} \frac{(-1)^{n+1}\, 4^n}{5^{n+1}} (x-2)^n$$

We multiply through and get 3 sums:

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+1} 4^{n+1}}{5^{n+1}} (x-2)^n + \sum_{n=0}^{\infty} \frac{(-1)^{n+1} 4^{n+1}}{5^{n+1}} (x-2)^{n+1} + \sum_{n=0}^{\infty} \frac{(-1)^{n+1} 4^n}{5^{n+1}} (x-2)^{n+2}$$

Next we want to make the powers of $(x-2)$ "match" so we can recombine the sums. So replace $n=m-2$ in the last sum, and $n=m-1$ in the middle, $n=m$ in the first.

$$\sum_{m=0}^{\infty} \frac{(-1)^{m+1} 4^{m+1}}{5^{m+1}} (x-2)^m + \sum_{m=1}^{\infty} \frac{(-1)^{m} 4^{m}}{5^{m}} (x-2)^m + \sum_{m=2}^{\infty} \frac{(-1)^{m-1} 4^{m-2}}{5^{m-1}} (x-2)^m$$

But the sums start at different m-values, so we treat the first couple terms separately:

<u>m=0 gives</u>: $\dfrac{(-1)4}{5} = \dfrac{-4}{5}$

<u>m=1 gives</u>: $\left(\dfrac{(-1)^2 4^2}{5^2} + \dfrac{(-1)4}{5}\right)(x-2) = \dfrac{-4}{25}(x-2)$

So we get

$$\frac{-4}{5} - \frac{4}{25}(x-2) + \underbrace{\sum_{m=2}^{\infty}\left(\frac{(-1)^{m+1}4^{m+1}}{5^{m+1}} + \frac{(-1)^m 4^m}{5^m} + \frac{(-1)^{m-1}4^{m-2}}{5^{m-1}}\right)}_{\text{simplify this if you want.}}(x-2)^m$$

for $-1 < -\frac{4}{5}(x-2) < 1$.

$$\Rightarrow \frac{5}{4} > (x-2) > -\frac{5}{4}$$

$$\Rightarrow \frac{13}{4} > x > \frac{3}{4}$$

Example: Find the Taylor series of $\ln(1+2x)$ about $x=0$.

Solution: The derivative of $\ln(1+2x)$ is $\frac{2}{1+2x}$.

So $\frac{2}{1+2x} = \frac{2}{1-(-2x)} = \frac{a}{1-x}$ with $a=2$ and x replaced by $-2x$.

So $$\frac{2}{1+2x} = \sum_{n=0}^{\infty} 2(-2x)^n = \sum_{n=0}^{\infty} 2^{n+1}(-1)^n x^n$$

for $-1 < -2x < 1$

$$\Rightarrow -\frac{1}{2} < x < \frac{1}{2}.$$

and then we integrate to go back!

$$\int \frac{2}{1+2x} = \sum_{n=0}^{\infty} \int (-1)^n 2^{n+1} x^n \, dx$$

$$\Rightarrow \ln(1+2x) = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{n+1}}{n+1} x^{n+1} + C,$$

plugging in the test number of $x=0$ gives $C=0$.

So $\ln(1+2x) = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{n+1}}{n+1} x^{n+1}$ for $-\frac{1}{2} < x < \frac{1}{2}$.