January 20 Tutorial 2.

Last day, we took a ton of limits of explicitly defined sequences, but no recursively defined ones. Here is one recursive one, just to see how it works in the cases where it's easy:

Example (§10.1 # 13)

Set $c_1 = 2$, $c_{n+1} = \dfrac{2}{c_n - 1}$, $n \geq 1$. What is the limit?

Solution: The only way we have to do a problem like this is to try and find an explicit formula for c_n, then take the limit.

We calculate the first few terms, with no (or a little) simplifying, which will help us see the pattern:

$$c_1 = 2$$

$$c_2 = \frac{2}{2-1} = \frac{2}{1} = 2$$

$$c_3 = \frac{2}{2-1} = 2 \ldots \quad \text{etc.}$$

So the limit is 2, because the explicit formula is

$c_n = 2$ for all n.

(§10.1, # 14)

Try $c_1 = 4$, $c_{n+1} = \dfrac{-c_n}{n^2}$, $n \geq 1$.

Then

$c_1 = 4$

$$c_2 = \frac{-4}{(1)^2} \quad (n=1)$$

$$c_3 = \overbrace{\frac{-4}{(1)^2}}^{c_2} \cdot \frac{-1}{(2)^2} \quad (n=2)$$

$$c_4 = \overbrace{\frac{-4}{(1)^2} \cdot \frac{-1}{(2)^2}}^{c_3} \cdot \frac{-1}{(3)^2} \quad (n=3)$$

In general, it looks like we get $c_{n+1} = \dfrac{4(-1)^n}{(n!)^2}$

So observe that

$$\frac{-4}{(n!)^2} \leq \frac{4(-1)^n}{(n!)^2} \leq \frac{4}{(n!)^2},$$ and taking limits as $n \to \infty$ we get

$$0 \leq \lim_{n\to\infty} \frac{4(-1)^n}{(n!)^2} \leq 0$$

so by the squeeze theorem, $\lim\limits_{n\to\infty} c_n = 0$.

Now transition to Taylor polynomials

~~Example~~

Remember that for a function $f(x)$, and a number c, the Taylor polynomial formula is:

$$P_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \dots + \frac{f^{(n)}(c)}{n!}(x-c)^n.$$

§10.3, #12. Find the Taylor series for

$$f(x) = \frac{1}{(1+3x)^2} \quad \text{about the point } c=0.$$

Plot enough Taylor polynomials to determine where the Taylor series converges to $f(x)$.

Solution: We need to take derivatives $f'(x), f''(x), f'''(x)$, to apply the Taylor polynomial formula.

So:

$$\frac{df}{dx} = \frac{-2}{(1+3x)^3} \cdot 3 \quad \text{(chain rule)}.$$

$$\frac{d^2 f}{dx^2} = -2 \cdot 3 \cdot \frac{d}{dx}\left(\frac{1}{(1+3x)^3}\right) = (-2)(-3) \cdot 3 \cdot 3 \cdot \frac{1}{(1+3x)^4}$$

$$\frac{d^3 f}{dx^3} = (-2)(-3) \cdot 3 \cdot 3 \, \frac{d}{dx}\left(\frac{1}{(1+3x)^4}\right) = (-2)(-3)(-4) \cdot 3 \cdot 3 \cdot 3 \, \frac{1}{(1+3x)^5}$$

In general,

$$f^{(n)}(x) = \frac{(-1)^n \cdot (n+1)! \cdot 3^n}{(1+3x)^{n+2}},$$

So we calculate ~~a few~~ the terms of the Talyor series:

$$\frac{f^{(n)}(c)}{n!} \overset{(c=0)}{=} \frac{(-1)^n \cdot (n+1)! \cdot 3^n}{(1+3\cdot 0)^{n+2} \cdot n!} = (-1)^n (n+1) \, 3^n$$

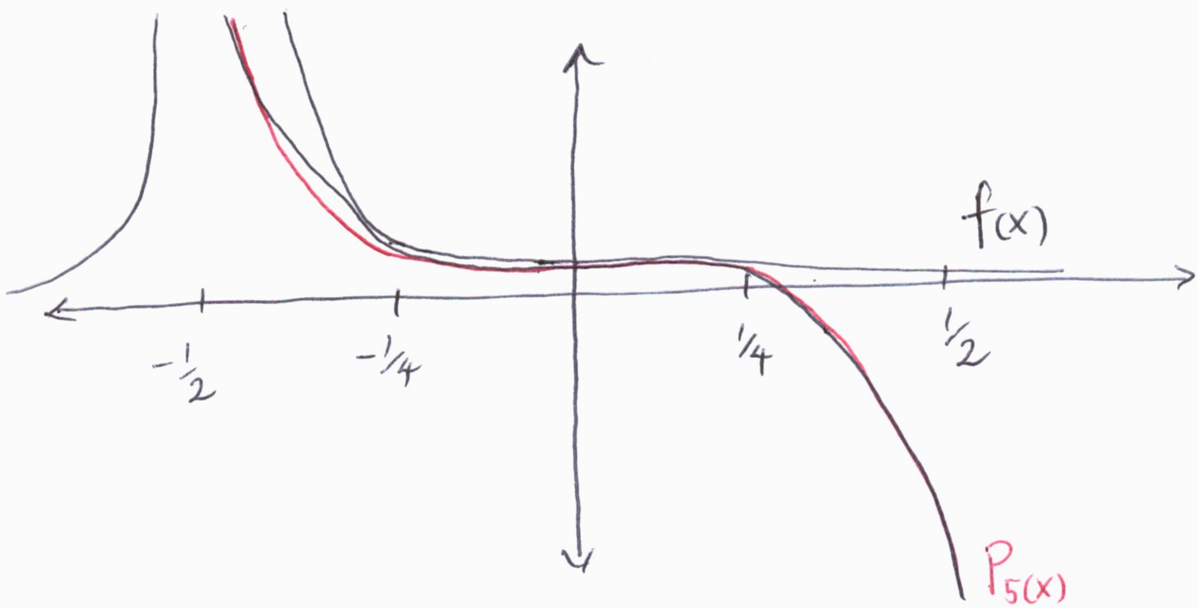
So the Taylor series is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n = \sum_{n=0}^{\infty} (-1)^n (n+1) 3^n \cdot x^n.$$

If we use Wolfram alpha to do some plotting, say using

$$P_5(x) = 1 - 6x + 27x^2 - 108x^3 + 405x^4 - 1458x^5$$

and $f(x) = \dfrac{1}{(1+3x)^2}$, then we get



So it looks like the series converges to $f(x)$ for x between... $-\frac{1}{4}$ and $\frac{1}{4}$? or $-\frac{1}{3}$ and $\frac{1}{3}$?

Next class we'll learn a calculation which allows us to calculate the <u>interval of convergence</u> of a series exactly.

Example: What is the Taylor series of $f(x) = \sqrt{1+3x}$ at $c=0$? Where does it converge? (Estimate graphically).

Solution: Again, we need a formula for the higher derivatives $f'(x)$, $f''(x)$, $f'''(x)$, ... etc.

Write $f(x) = (1+3x)^{1/2}$ and calculate

$$f'(x) = \frac{1}{2}(1+3x)^{-1/2} \cdot 3$$

$$f''(x) = \frac{1}{2} \cdot \frac{-1}{2} \cdot (1+3x)^{-3/2} \cdot 3 \cdot 3$$

$$f'''(x) = \frac{1}{2} \cdot \frac{-1}{2} \cdot \frac{-3}{2} \cdot 3 \cdot 3 \cdot 3 \cdot (1+3x)^{-5/2}$$

$$f^{(4)}(x) = \frac{1}{2} \cdot \frac{-1}{2} \cdot \frac{-3}{2} \cdot \frac{-5}{2} \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot (1+3x)^{-7/2}.$$

So in general

$$f^{(n)}(x) = (-1)^{n-1} \cdot \frac{1}{2^n} \cdot (1)(3)(5)\underset{(2n+3)}{\cdots}\, 3^n (1+3x)^{-\frac{2n-1}{2}}$$

cleaning up

$$\frac{(-1)^{n-1} 3^n (1)(3)(5) \cdots (2n-3)\,(1+3x)^{-\frac{2n-1}{2}}}{2^n}$$

Now what $(1)(3)(5) \cdots (2n-3)$ is meant to represent is the product of the first $n-1$ odd numbers. Leaving it in this form is <u>fine</u>. There is a lesser-know shorthand for this: $(2n-3)!!$ (double factorial).

So for the Taylor series, we get

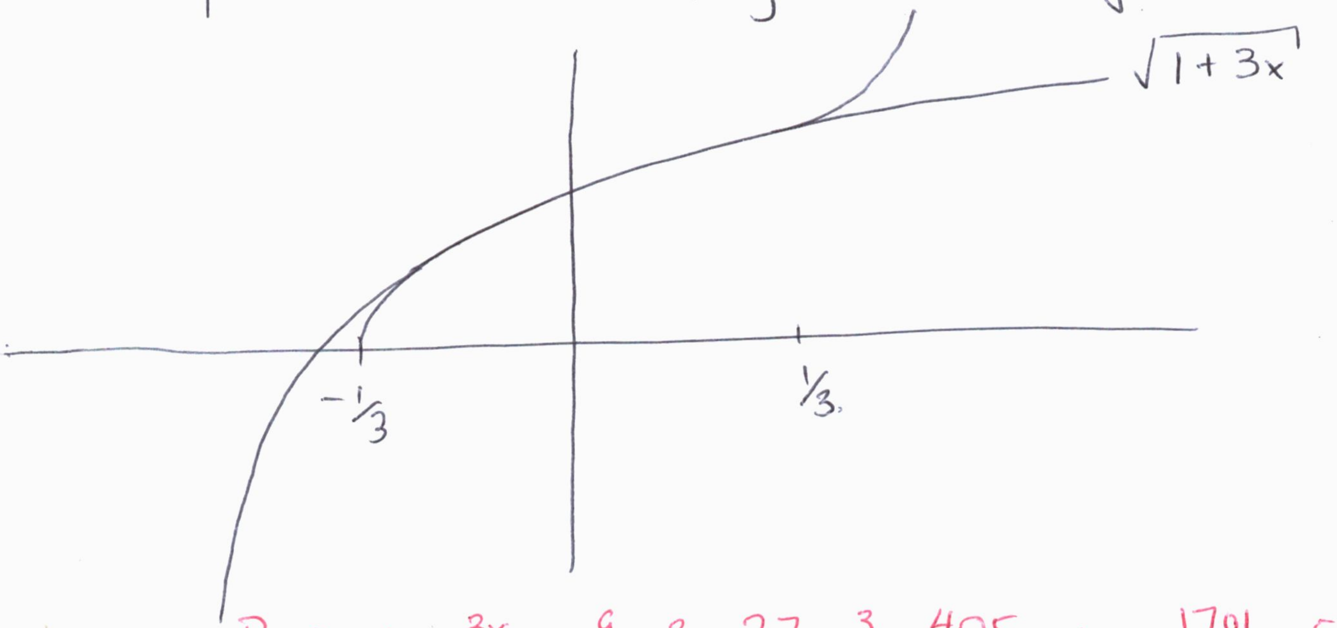
$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$$

take $c=0$, then $(1+3c)^{\frac{-2n-1}{2}} = 1$

$(x-c)^n = x^n$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \cdot \frac{(-1)^{n-1} 3^n (2n-3)!!}{2^n} x^n = \sqrt{1+3x}$$

If we plot the first few polynomials, we get



$$P_5(x) = 1 + \frac{3x}{2} - \frac{9}{8}x^2 + \frac{27}{16}x^3 - \frac{405}{128}x^4 + \frac{1701}{256}x^5$$

So it looks like we have convergence for x between $-\frac{1}{3}$ and $\frac{1}{3}$. (This turns out to be true, we'll see this tomorrow).

Example: Calculate the Taylor series of $\tan^{-1}(x)$ at $c=0$. Estimate graphically where it converges. What does setting $x=1$ give?

Solution: As before, we need to calculate derivatives of $f(x) = \tan^{-1}(x)$:

$$f'(x) = \frac{1}{x^2+1} = (x^2+1)^{-1}$$

$$f''(x) = -1 \cdot (x^2+1)^{-2} \cdot 2x$$

$$f'''(x) = \frac{(6x^2-2)}{(x^2+1)^3}$$

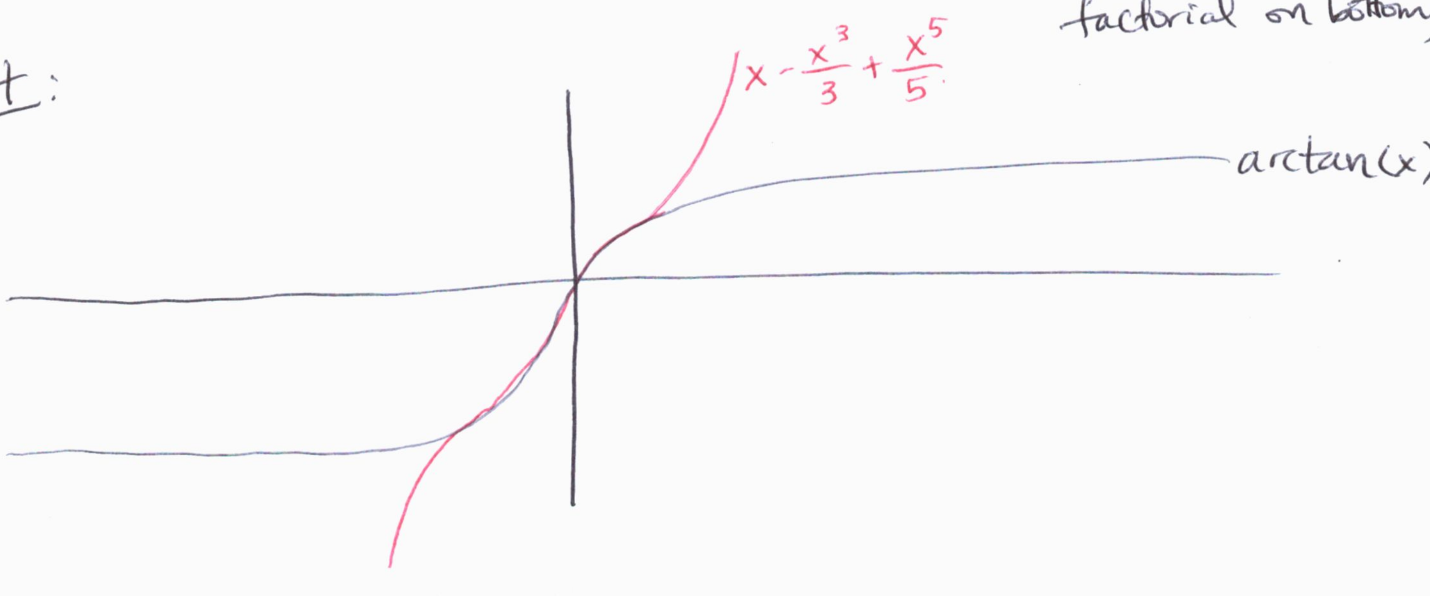
$$f^{(4)}(x) = \frac{24x(x^2-1)}{(x^2+1)^4}$$, and in general, the formula is too difficult!!

However, we'll learn a trick to get around this difficulty, and we'll get

$$\tan^{-1}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$$

$$= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$ (looks like $\sin(x)$ but no factorial on bottom)

Plot:



and taking more complicated polynomials does not improve the convergence: It converges for x between -1 and 1.

With a bit of work we can also show it converges at $x=1$, so we get

$$\tan^{-1}(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$$

ie. $$\pi/4 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} \dots$$ which is kind of neat.

<u>Example</u>: <u>§10.1, #10</u>.

Find the Taylor series of $f(x) = \frac{1}{2-x}$, and estimate where it converges.

<u>Solution</u>: We can compute derivatives as before, or we can use a formula we learned from section 10.4:

$$\sum_{n=0}^{\infty} ax^n = \frac{a}{1-x} \quad \text{if } |x|<1.$$

So, can we make $f(x) = \frac{1}{2-x}$ look like $\frac{a}{1-x}$? Yes! Divide top and bottom by 2:

$$f(x) = \frac{1}{2-x} = \frac{\frac{1}{2}}{1-\left(\frac{x}{2}\right)}$$

Set $u = \frac{x}{2}$, then it's $\dfrac{\left(\frac{1}{2}\right)}{1-u}$, which has

Taylor series

$$\sum_{n=0}^{\infty} \frac{1}{2} u^n = \frac{\frac{1}{2}}{1-u} \quad \text{for } |u| < 1.$$

This gives, setting $u = \frac{x}{2}$.

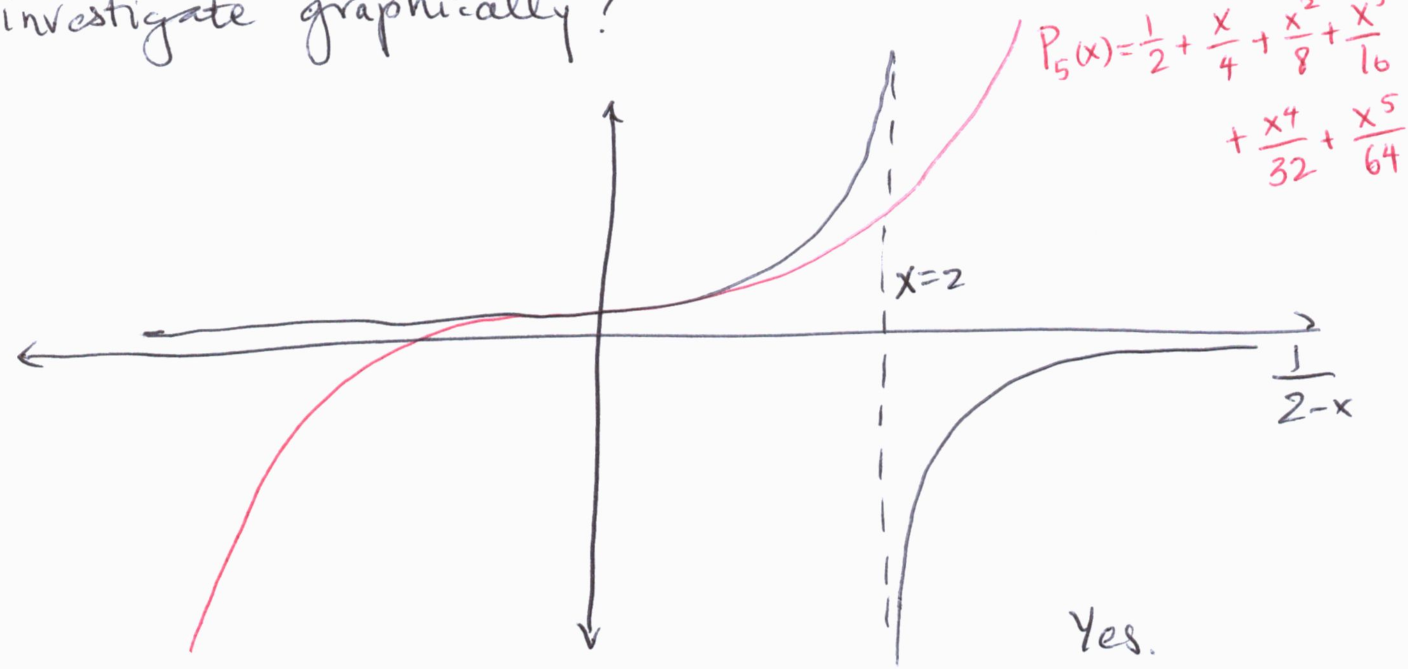
$$\frac{\frac{1}{2}}{1-\frac{x}{2}} = \sum_{n=0}^{\infty} \frac{1}{2}\left(\frac{x}{2}\right)^n = \sum_{n=0}^{\infty} \frac{x^n}{2^{n+1}} \quad \text{for } \left|\frac{x}{2}\right| < 1$$

$$\parallel$$

$$f(x)$$

But $\left|\frac{x}{2}\right| < 1 \iff |x| < 2.$

Does our calculation agree with what we'll find if we investigate graphically?



Yes.