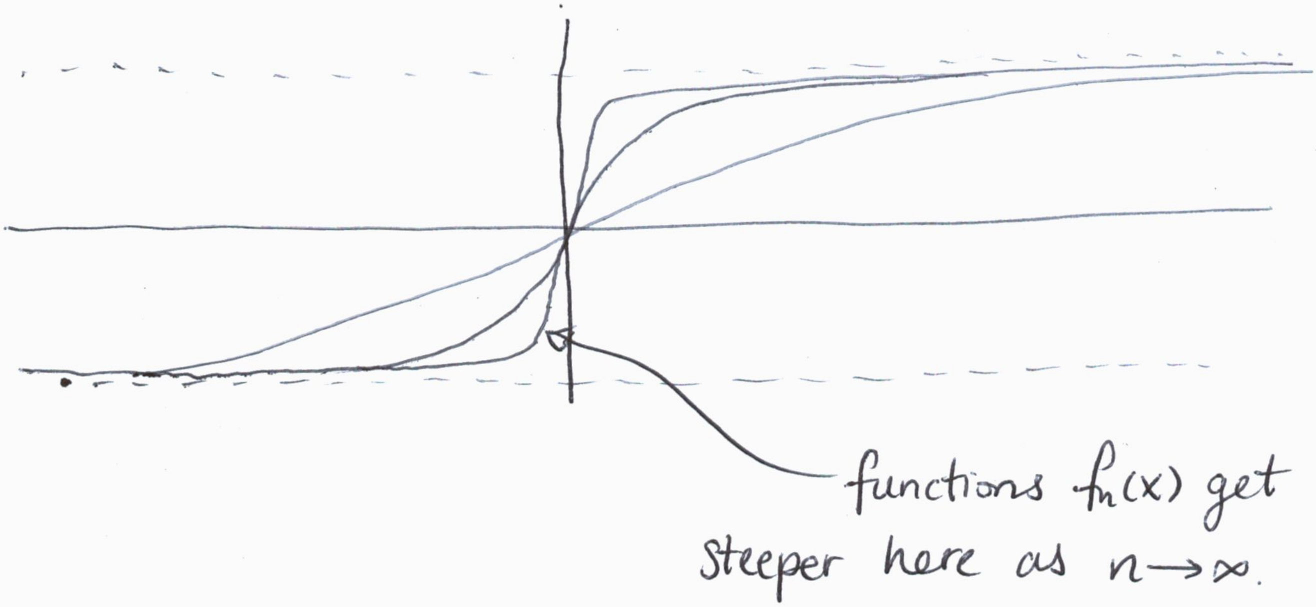
Tutorial 10

This tutorial will be the beginning of a review that will continue in Friday's lecture. Since the final exam is comprehensive, we'll start at the beginning with power series.

Recall we say that a sequence of functions $\{f_n(x)\}_{n=1}^{\infty}$ <u>converges</u> to a limit function $f(x)$ if $\lim_{n\to\infty} f_n(x) = f(x)$ for all x.

<u>Example</u>: Our "classic example" is

$$f_n(x) = \tan^{-1}(nx).$$



Then $\lim_{n\to\infty} \tan^{-1}(nx) = \begin{cases} \frac{\pi}{2} & \text{if } x>0 \\ 0 & \text{if } x=0 \\ -\frac{\pi}{2} & \text{if } x<0 \end{cases}$

since the function $\tan^{-1}(x)$ has horizontal asymptotes at $\pm\pi/2$. In particular,

$$\lim_{x\to-\infty} \tan^{-1}(x) = -\pi/2 \quad \text{and} \quad \lim_{x\to\infty} \tan^{-1}(x) = \pi/2 .$$

From a given function $f(x)$, we can calculate its Taylor polynomials $P_n(x)$, each comes with a remainder $R_n(x)$ at c so that

$$f(x) = P_n(x) + R_n(x)$$

The formulas are:

$$P_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n$$

and

$$R_n(x) = \frac{f^{(n+1)}(z_n)}{(n+1)!}(x-c)^{n+1}, \quad z_n \text{ between } x \text{ and } c.$$

Then $\lim\limits_{n\to\infty} P_n(x) = f(x)$ as long as $\lim\limits_{n\to\infty} R_n(x) = 0$.

<u>Example</u>: Calculate the Taylor polynomials for $n = 0, 1, 2, 3$ centered at $c = 3$ for the function $f(x) = \frac{1}{x-2}$.

<u>Solution</u>: If $n=0$ then

$$P_0(x) = f(c) = \frac{1}{3-2} = 1$$

Then we need derivatives for the other polynomials:

$$f'(x) = \frac{-1}{(x-2)^2}, \quad f''(x) = \frac{2}{(x-2)^3}, \quad f'''(x) = \frac{-6}{(x-2)^4}$$

So

$$f'(3) = -1, \quad f''(3) = 2, \quad f'''(3) = -6.$$

So

$$P_1(x) = 1 + (-1)(x-3)$$

$$P_2(x) = 1 + (-1)(x-3) + \frac{2}{2!}(x-3)^2$$

$$P_3(x) = 1 + (-1)(x-3) + \frac{2}{2!}(x-3)^2 + \frac{(-6)}{3!}(x-3)^3$$

Recall we also had a lot of terminology:

The <u>Maclaurin series</u> of a function f is the Taylor series centered at 0.

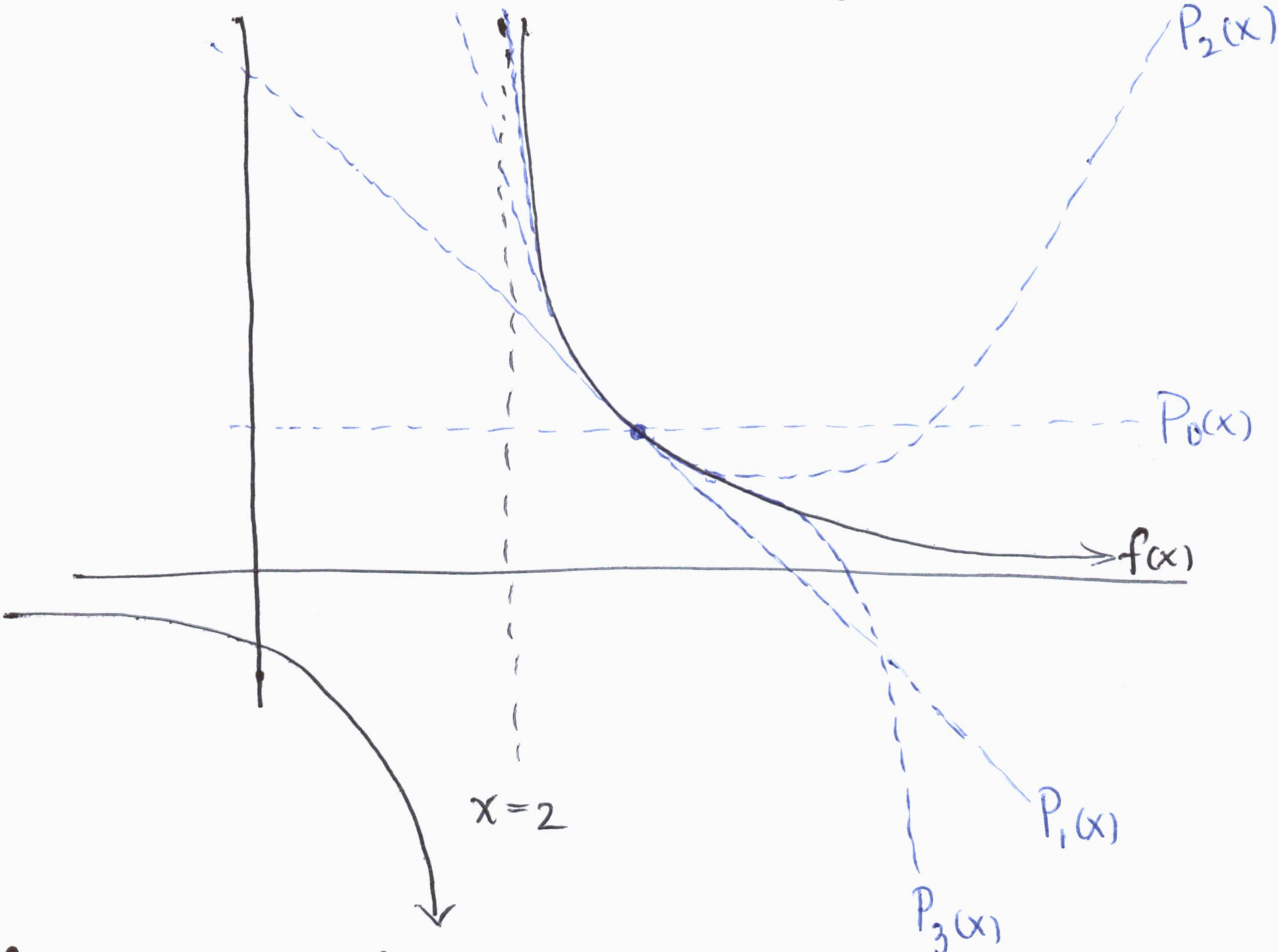
Example: # 10 §10.3 (slight change)

Consider $f(x) = \frac{+1}{x-2}$, with $c=3$. Plot enough Taylor polynomials to determine the interval of convergence and the series.

From the computations we just did, we arrive at

$$\frac{1}{x-2} = \sum_{n=0}^{\infty} (-1)^n (x-3)^n.$$

If we plot the first few polynomials, we get:



Get convergence for $2 < x < 4$.

A few "famous" power series are needed in order to help us ① figure out the power series of other functions and ② take sums of given power series. The 'famous' formulas are:

$$\sum_{n=0}^{\infty} a x^n = \frac{a}{1-x} \qquad -1 < x < 1 \quad \left(\begin{matrix}\text{geometric}\\ \text{series}\end{matrix}\right).$$

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x \qquad \text{for all } x, \text{ also}$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n, \quad -1<x<1.$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = \cos(x) \quad \text{for all } x.$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)!} x^{2n-1} = \sin(x) \quad \text{for all } x.$$

Moreover, when we come up with a new power series formula we always need to give its <u>radius of convergence</u> to say where the formula is valid.

This number is calculated for a power series $\sum_{n=0}^{\infty} a_n x^n$ by:

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| \quad \text{or} \quad R = \lim_{n\to\infty} \frac{1}{\sqrt[n]{|a_n|}}$$

Not only do we have algebra tricks that we can do on power series in order to analyze them, but we can also integrate/differentiate them without changing the radius of convergence.

<u>Example:</u> Find the sum and radius of convergence of $\sum_{n=0}^{\infty} n(n+1)x^n$.

<u>Solution:</u> We can first integrate to get some good cancelation:

$$S(x) = \sum_{n=0}^{\infty} n(n+1)\, x^n$$

$$\Rightarrow \int S(x)\,dx = \sum_{n=0}^{\infty} \int n(n+1)\, x^n\,dx = \sum_{n=0}^{\infty} n\, x^{n+1}$$

Now

$$\sum_{n=0}^{\infty} n\,x^{n+1} = x^2 \sum_{n=0}^{\infty} n\,x^{n-1}$$

$$= x^2 \frac{d}{dx}\left(\sum_{n=0}^{\infty} x^n\right) = x^2 \frac{d}{dx}\left(\frac{1}{1-x}\right)$$

if $-1<x<1$.

$$= x^2\left(\frac{1}{(1-x)^2}\right)$$

$$\Rightarrow \int S(x)\,dx = \frac{x^2}{(1-x)^2},$$ so differentiate to get back to $S(x)$

$$\Rightarrow S(x) = \frac{-2x}{(x-1)^3}, \quad \underbrace{-1 < x < 1.}$$

the interval is the same as the interval for the formula $\sum_{i}^{\infty} x^n = \frac{1}{1-x}$ that we used.

When completely stuck, remember we also have the <u>binomial formula</u>:

$$(x+1)^m = 1 + \sum_{n=1}^{\infty} \frac{m(m-1)(m-2)\cdots(m-n+1)}{n!} x^n, \quad -1 < x < 1$$

which can come to the rescue when things are looking particularly bad.

Also recall that we can add, multiply, subtract and divide power series. In this case the radius of convergence of the sum/difference/product is the smaller of the two radii that you started with. Note that in order to do any operation to two power series, they must both be centred at the same value.