Tutorial 1.

This tutorial is basically a review of all the classic limit-taking techniques from 1500 and 1700, but now we are applying them to sequences instead of functions.

Example: Consider a sequence defined by

$$c_n = \frac{3n^3-1}{2n^3+1}.$$

Does the sequence converge? If so, what is the limit?

Solution: This sequence converges when the terms approch some number L as $n \to \infty$. In other words, we need to take the limit:

$$\lim_{n\to\infty} \frac{3n^3-1}{2n^3+1} = \lim_{n\to\infty} \frac{\frac{1}{n^3}(3n^3-1)}{\frac{1}{n^3}(2n^3+1)}$$

$$= \lim_{n\to\infty} \frac{3-\cancel{\frac{1}{n^3}}^{\to 0}}{2+\cancel{\frac{1}{n^3}}^{\to 0}} = \frac{3}{2}$$

Example: Does the sequence $c_n = \frac{(-1)^n \cdot n}{n+1}$ converge?

If yes, what is the limit?

Solution: If we try to take the limit

using the same trick:

$$\lim_{n\to\infty} \frac{(-1)^n \cdot n}{n+1} \cdot \frac{\frac{1}{n}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{\frac{(-1)^n \cdot \cancel{n}}{\cancel{n}}}{1+\frac{1}{n}}$$

$$= \lim_{n\to\infty} \frac{(-1)^n}{1+\frac{1}{n}}$$

we don't get a numerator which converges.
In fact, for even values of n (ie when $n=2k$)
we get

$$= \lim_{k\to\infty} \frac{(-1)^{2k}}{1+\frac{1}{2k}} = \lim_{k\to\infty} \frac{1}{1+0} = 1$$

and for odd we get (ie $n=2k+1$)

$$= \lim_{k\to\infty} \frac{(-1)^{2k+1}}{1+\frac{1}{2k+1}} = \lim_{k\to\infty} \frac{-1}{1+0} = -1$$

So as $n\to\infty$, even-numbered terms approach 1, odd-numbered terms approach -1. Therefore this sequence cannot converge.

Remark: To show a sequence doesn't converge, it is enough to argue that the terms in the sequence "stay separated" as $n\to\infty$, as we argued here.

Example: What is the limit of the sequence

$$c_n = \sqrt{n^2+1} - n, \quad \text{if it exists?}$$

Solution: Here, the trick is to multiply by a certain quotient:

$$\left(\sqrt{n^2+1} - n\right) \cdot \frac{\sqrt{n^2+1} + n}{\sqrt{n^2+1} + n}$$

$$= \frac{\cancel{n^2}+1-\cancel{n^2}}{\sqrt{n^2+1}+n} = \frac{1}{\sqrt{n^2+1}+n}$$

So $\displaystyle\lim_{n\to\infty} \sqrt{n^2+1} - n = \lim_{n\to\infty} \frac{1}{\sqrt{n^2+1}+n} = 0$, since the bottom clearly goes to ∞.

Example: Find the limit of the sequence

$$c_n = \left(\tan^{-1}\left(\tfrac{1}{n}\right)\right)^{1/n} \quad \text{(it exists!)}.$$

Solution: The limit

$$\lim_{n\to\infty} \left(\tan^{-1}\left(\tfrac{1}{n}\right)\right)^{1/n} \quad \text{becomes}$$

$$\lim_{x\to 0} \left(\tan^{-1}(x)\right)^{x} \quad \text{upon substituting } x = \tfrac{1}{n}$$

and noting that $x \to 0$ as $n \to \infty$.

Then recall that $e^{\ln(x)} = x$, so we can write

$$(\tan^{-1}(x))^x = e^{\ln((\tan^{-1}(x))^x)}$$
$$= \exp(x \ln(\tan^{-1}(x)))$$

So we're interested in

$$\lim_{x\to 0} \exp(x \ln(\tan^{-1}(x))).$$

Recall that limits can be brought inside of continuous functions, so this limit is

$$= \exp\left(\lim_{x\to 0} x \ln(\tan^{-1}(x))\right)$$ since the exponential is continuous.

Rewrite

$$x \ln(\tan^{-1}(x)) = \frac{\ln(\tan^{-1}(x))}{\frac{1}{x}},$$

then note the top and bottom both go to ∞ as $x \to 0$.

So we can apply L'Hôpital's rule: $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$

Replacing top and bottom with their derivatives, we get

$$\lim_{x\to 0} \frac{\ln(\tan^{-1}(x))}{\frac{1}{x}} = \lim_{x\to 0} \left(\frac{\frac{1}{x^2 \tan^{-1}(x) + \tan^{-1}(x)}}{-\frac{1}{x^2}} \right)$$

$$= \lim_{x\to 0} \frac{-x^2}{(x^2+1)\tan^{-1}(x)}.$$

Now top and bottom both go to 0 as $x \to 0$, so we can apply L'Hôpital's rule again:

$$= \lim_{x\to 0} \frac{-2x}{\dfrac{-2x\tan^{-1}(x) - 1}{(x^2+1)^2\left(\tan^{-1}(x)\right)^2}}$$

$$= \lim_{x\to 0} \frac{-2x(x^2+1)^2\left(\tan^{-1}(x)\right)^2}{-2x\tan^{-1}(x) - 1} = \frac{0}{1} = 0.$$

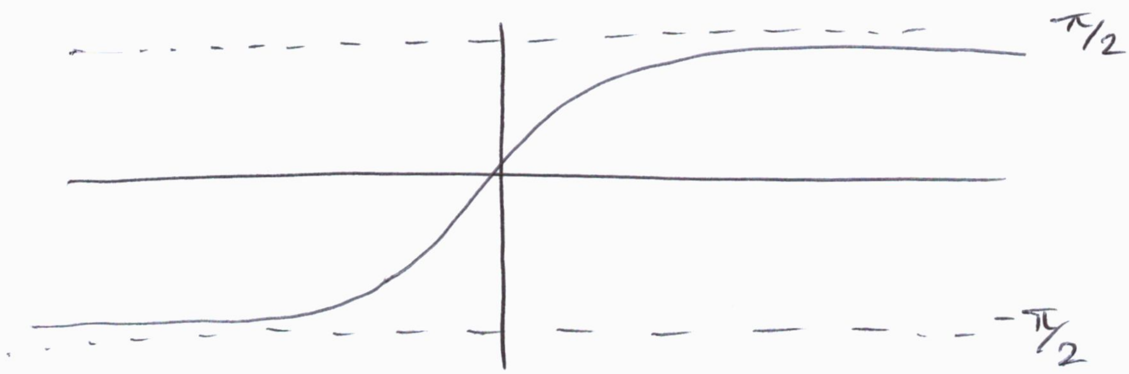
Now the top goes to 0, bottom to 1, since $\lim_{x\to 0} \arctan(x) = 0$.

Thus $\lim_{x\to 0} \exp(\text{stuff}) = \cancel{\lim}\ e^0 = 1.$

<u>Example</u>: Find the limit of the sequence

$$c_n = \frac{\tan^{-1}(n)}{n} \quad \text{as } n \to \infty \text{ if it exists.}$$

<u>Solution</u>: Recall that $\tan^{-1}(x)$ looks like:



So $\lim_{x\to\infty} \tan^{-1}(x) = \pi/2$, and $\tan^{-1}(n) \geq 0 \quad \forall n > 0$.

Then using the squeeze theorem, we get

$$0 \leq \tan^{-1}(n) \leq \frac{\pi}{2}$$

$$\Rightarrow \quad 0 \leq \tan^{-1}(n) \leq \frac{\pi}{2n}$$

$$\Rightarrow \quad 0 \leq \lim_{n\to\infty} \frac{\tan^{-1}(n)}{n} \leq \lim_{n\to\infty} \frac{\pi}{2n} = 0$$

$\Rightarrow$ the limit exists and is zero.

Note that I'm only doing explicitly defined sequences instead of recursive ones. Recursive sequences are simply too hard most of the time, e.g.

Example: Set $a_1 = N$, a large positive integer.

Compute successive terms by doing:

$$a_{n+1} = \begin{cases} a_n/2 & \text{if } a_n \text{ is even} \\ 3a_n + 1 & \text{if } a_n \text{ is odd} \end{cases}$$

and $a_{n+1} = 1$ if $a_n = 1$. (So once you hit a 1, do ones forever after that point).

Show $\lim_{n\to\infty} a_n = 1$. This is considered one of the hardest problems in mathematics and is known as the Collatz Conjecture.

Example: What is the limit of the sequence

$$c_n = \sqrt{\left(1 + \frac{1}{2n}\right)^n}\,, \quad \text{if it exists?}$$

Solution: There are some famous limits that you should know, like

$$\lim_{x\to 0} \frac{\sin x}{x} = 1\,, \quad \text{and} \quad \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e.$$

Obviously for this sequence we want to use the second limit somehow. So we rewrite

$$\lim_{n\to\infty}\sqrt{\left(1+\frac{1}{2n}\right)^n} = \sqrt{\lim_{n\to\infty}\left[\left(1+\frac{1}{2n}\right)^{2n}\right]^{\frac{1}{2}}}$$

$$= \sqrt{\lim_{n\to\infty}\sqrt{\left(1+\frac{1}{2n}\right)^{2n}}}$$

$$= \sqrt{\sqrt{\lim_{n\to\infty}\left(1+\frac{1}{2n}\right)^{2n}}}$$

$$= \sqrt{\sqrt{e}} = \left(e^{\frac{1}{2}}\right)^{\frac{1}{2}} = e^{1/4} \checkmark$$

Example: What is the limit of the sequence of functions $f_n(x) = \tan^{-1}(nx)$?

So the limit function is:

$$f(x) = \begin{cases} \pi/2 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -\pi/2 & \text{if } x < 0. \end{cases}$$

Example: Does the sequence of functions

$$f_n(x) = \frac{nx^2+1}{nx-1}$$

converge to a limit function $f(x)$? If so, what is $f(x)$ and where is it defined?

Solution: We need to know if $\lim_{n\to\infty} f_n(x)$ exists for different values of x.

We take limits:

$$\lim_{n\to\infty} \frac{nx^2+1}{nx-1} = \lim_{n\to\infty} \frac{\frac{1}{n}(nx^2+1)}{\frac{1}{n}(nx-1)}$$

$$= \lim_{n\to\infty} \frac{x^2+\frac{1}{n}}{x-\frac{1}{2}}$$

$$= \frac{x^2}{x} = x.$$

and this limit works no matter the value of x.

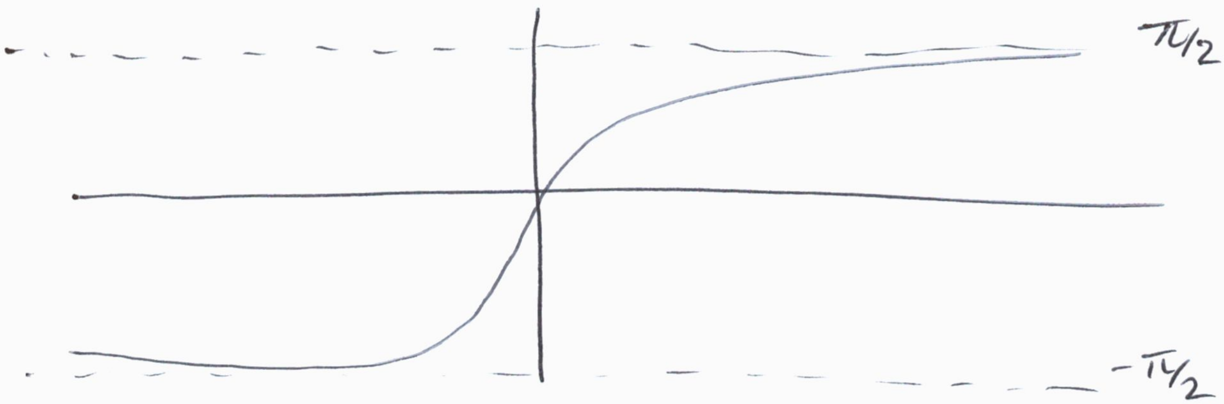
So $f(x) = x$ is the limit

Solution:

For each real number x, the limit function $f(x)$ is given by

$$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \tan^{-1}(nx)$$

Recall that $\tan^{-1}(x)$ has graph:



And so we're going to get 3 possibilities:

① If $x<0$ then

$$\lim_{n\to\infty} \tan^{-1}(nx) = \lim_{x\to -\infty} \tan^{-1}(x) = -\pi/2$$

② If $x>0$ then

$$\lim_{n\to\infty} \tan^{-1}(nx) = \lim_{x\to \infty} \tan^{-1}(x) = \pi/2$$

③ If $x=0$ then $\tan^{-1}(nx) = \tan^{-1}(0) = 0$ for every n.