The holonomy correspondence I

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Surface group representations

Definition: Let G be a group. A representation of G is a homomorphism $\rho : G \longrightarrow GL(V)$, ie. V gets a linear G-action.

Fix an identification $V = \mathbb{C}^n$.

Q: When are two representations $\rho_1, \rho_2 : G \longrightarrow GL_n(\mathbb{C})$ the same? (Equivalent)

Ans: We say $\rho_1 \sim \rho_2$ whenever there exists a change of basis $\varphi : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ such that

$$\begin{array}{ccc} \mathbb{C}^n & \xrightarrow{\varphi} & \mathbb{C}^n \\ {\scriptstyle \rho_1(g)}\downarrow & & \downarrow{\scriptstyle \rho_2(g)} \\ \mathbb{C}^n & \xrightarrow{\varphi} & \mathbb{C}^n \end{array}$$

commutes for every $g \in G$. Alternatively, if we think of $\varphi \in GL_n(\mathbb{C})$ then $\rho_2 = \varphi \rho_1 \varphi^{-1}$

So we have a $GL_n(\mathbb{C})$-action on the space of all homomorphisms $\mathrm{Hom}(G, GL_n(\mathbb{C}))$ by conjugation.

Set

$$M(G) = \mathrm{Hom}(G, GL_n(\mathbb{C}))/\sim$$

"space of representations of a fixed dimension"

As an example:

Set $G = \mathbb{Z}$. Then any $\rho : G \to GL_n(\mathbb{C})$ is determined by $\rho(1) \in GL_n(\mathbb{C})$, and all equivalent representations would send 1 to an element conjugate to $\rho(1)$.

So
$$M(G) = \{\text{conjugacy classes in } GL_n(\mathbb{C})\}$$

E.g. Take $n=2$, use Jordan normal form to get a "better description" of $M(G)$: Think of the conjugacy classes corresponding to

$$\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_2 \end{bmatrix}$$

So you get $M(G) \simeq (\mathbb{C}^\times \times \mathbb{C}^\times)/\mathbb{Z}_2 \cup \mathbb{C}^\times$, since every conjugacy class is either a pair of eigenvalues or a single repeated eigenvalue.

<u>Example</u>: Take $G = \pi_1(S^1 \times S^1) \simeq \mathbb{Z} \times \mathbb{Z}$, then any $\rho : G \to GL_n(\mathbb{C})$ is determined by $\rho(1,0)$ and $\rho(0,1)$. Then

$$M(G) = \{(a,b) \in GL_n(\mathbb{C}) \times GL_n(\mathbb{C}) \mid aba^{-1}b^{-1} = id\} \Big/ GL_n(\mathbb{C})$$

This is a special case of $M(\Sigma) = M(\pi_1(\Sigma))$
= moduli space of surface group representations.

Another generalization: $M_K(B) = \mathrm{Hom}(\pi_1(B), K)/K$

Lie group   manifold   conjugation by elements of K

Some basic questions about these spaces are tough, e.g. what's the number of connected components?

Flat Bundles.

Def: A fibre bundle is (E, B, π, F) consisting of a smooth map $\pi : E \longrightarrow B$ such that for each $b \in B$ $\exists$ an open nbhd U and a diffeomorphism

$h : U \times F \longrightarrow \pi^{-1}(U)$ such that $\pi \circ h = pr_1$

$$\begin{tikzcd} U \times F \arrow[rr, "h"] \arrow[dr, "pr_1"'] & & \pi^{-1}(U) \arrow[dl, "\pi"] \\ & U & \end{tikzcd}$$

$E :=$ Total space
$B :=$ Base space
$F :=$ Fibre

Examples:
- $E = B \times F$, $\pi = pr_1$, $U = B$ is a trivial example.
- $E = TB$, the tangent bundle of the base space B (projection π is to send each tangent space to its corresponding point)

Note: Locally fibre bundles ~~are~~ are "trivial", but globally there could be "twisting", e.g. the Möbius band 

Now we want to consider a special bundle type.

<u>Definition</u>: A principal G-bundle is (E, B, π, G) a fibre bundle, where each fibre $E_b = \pi^{-1}(b)$ carries a free and transitive G-action, and the maps

$$h : U \times G \longrightarrow \pi^{-1}(U) \text{ are "}G\text{-equivariant"}$$

$$h(b, ga) = h(b,g)\underset{\text{action}}{\cdot} a$$

<u>Examples</u>: • $E = B \times G$

• $E = Fr(TB)$ (the frame bundle)

$= \{(v_1, \dots, v_n) \text{ ordered bases of } TB\}$, and bases are related to one another by change of basis matrices,

so it's a $GL_n(\mathbb{R})$-bundle.

• $$\begin{tikzcd} E \arrow[d] & = & S^1 \arrow[d] & z \arrow[d, maps to] \\ B & & S^1 & z^k \end{tikzcd}$$ is a principal $\mathbb{Z}/\mathbb{Z}_k$-bundle.

<u>Q</u>/ When are two principal G-bundles (over B) equivalent?

<u>Ans</u>: We say $E_1 \sim E_2$ if and only if there exists a diffeomorphism $\varphi : E_1 \longrightarrow E_2$ such that

$$\begin{tikzcd} E_1 \arrow[rr, "\varphi"] \arrow[dr, "\pi_1"'] & & E_2 \arrow[dl, "\pi_2"] \\ & B & \end{tikzcd}$$

commutes, and such that

φ is G-equivariant: $\varphi(e \cdot g) = \varphi(e) \cdot g$

Connections

Definition: A connection on a principal G-bundle (E, B, π, G) is a smooth assignment $p \longmapsto \mathcal{H}_p$ of a subspace $\mathcal{H}_p \subset T_p E$ satisfying

1. $T_p E = \mathcal{H}_p \oplus T_p E_b$
2. $\mathcal{H}$ is G-invariant, meaning $\mathcal{H}_{p \cdot g} = R_{g*} \mathcal{H}_p$

this is the "right action by g" map.

Note: $\mathcal{H}_p$ is always isomorphic to the tangent space $T_{\pi(p)} B$, by an application of the rank-nullity theorem.

What are connections good for?

- They allow for path-lifting.

For any $\gamma : [a,b] \longrightarrow B$ and any $p \in E$ with $\pi(p) = a$, there exists a unique path $\tilde{\gamma} : [a,b] \longrightarrow E$ with $\frac{d\tilde{\gamma}}{dt}(t) \in \mathcal{H}_{\tilde{\gamma}(t)}$. This happens because of $\mathcal{H}_p \simeq T_{\pi(p)} B$, so any tangent vector in $T_{\pi(p)} B$ arising from $\frac{d\gamma}{dt}$ already has a specified ~~tangent~~ horizontal direction arising from the identification $\mathcal{H}_p \simeq T_{\pi(p)} B$