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The coadjoint representation

Recall from last time: A Lie group is a manifold with a smooth group operation.

E.g. $GL(n,\mathbb{R})$, $SL(n,\mathbb{R})$, $O(n) = \{A \mid A^TA = 1\}$

$$SO(n) = \{A \mid A^TA = 1,\ \det(A) = 1\}$$

Also to every G, we have an associated Lie algebra $\mathfrak{g} = \left\{ \frac{d}{dt}\Big|_0 A(t) \;\middle|\; \begin{matrix} A(t) \in G \\ A(0) = 1 \end{matrix} \right\}$. It's a vector space with bracket $\mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ given by

- $[b,a] = -[a,b]$
- $[a,[b,c]] + [\text{cyclic permutations}] = 0$
- linear in coordinates.

E.g. $\mathfrak{gl}(n,\mathbb{R})$, $\mathfrak{sl}(n,\mathbb{R}) = \{A \mid tr(A) = 0\}$

$\mathfrak{o}(n) = \{a \mid a^T + a = 0\}$

$\mathfrak{so}(n) = \{a \mid a^T + a = 0,\ tr(a) = 0\}$.

For matrix groups, the bracket is always $[A,B] = AB - BA$

We ended last day with the adjoint map

$$Ad : G \longrightarrow GL(\mathfrak{g})$$

$$Ad_g(a) = \frac{d}{dt}\Big|_{t=0} g\exp(ta)g^{-1}.$$

Def: The coadjoint representation $Ad_g^*: \mathfrak{g}^* \longrightarrow \mathfrak{g}^*$ is defined by $\langle Ad_g^* \zeta, a \rangle = \langle \zeta, Ad_{g^{-1}} a \rangle$ inverse here

Note (i) $\langle \ , \ \rangle$ is evaluation of a linear function on an element of the space.

(ii) We'll identify $\mathfrak{g} \simeq \mathbb{R}^n$, so $\mathfrak{g}^* \simeq (\mathbb{R}^n)^* \simeq \mathbb{R}^*$. So we're able to identify $\mathfrak{g}$ with columns and $\mathfrak{g}^*$ with rows, and the pairing becomes matrix multiplication.

Example 1: $G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \middle| \ a \neq 0 \right\}$, affine transformations of $\mathbb{R}$.

Then $\mathfrak{g} = \left\{ \begin{pmatrix} u & v \\ 0 & 0 \end{pmatrix} \right\} \cong \mathbb{R}^2$, and the adjoint representation is

$$Ad_{(a,b)} \begin{pmatrix} u & v \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u & v \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1/a & -b/a \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} u & av - ub \\ 0 & 0 \end{pmatrix} =$$

Rewrite as $Ad_{(a,b)} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -b & a \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u \\ av - ub \end{pmatrix}$

(Recall for matrix groups $Ad_g(a) = gag^{-1}$).

Then the coadjoint representation is

$$\langle Ad_{(a,b)}^* [\zeta, \eta], \begin{bmatrix} u \\ v \end{bmatrix} \rangle = [\zeta, \eta] \begin{bmatrix} 1 & 0 \\ -b & a \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

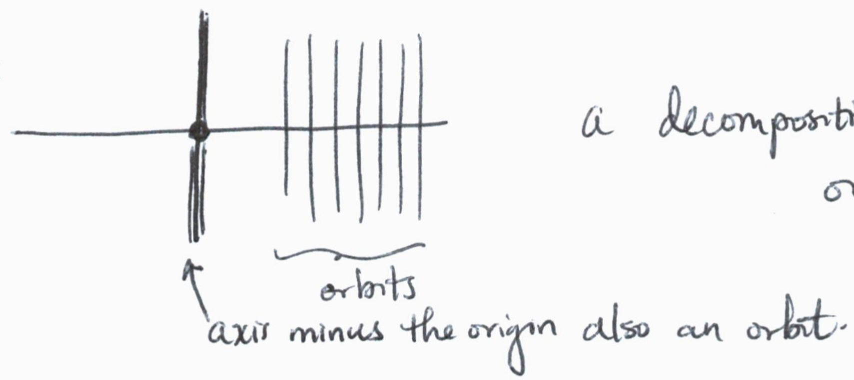
Therefore $Ad^*_{(a,b)}[\zeta, \eta] = [\zeta, \eta]\begin{bmatrix} 1 & 0 \\ -b & a \end{bmatrix}$

$$= [\zeta - \eta b \quad \eta a].$$

What of adjoint and coadjoint orbits?

- adjoint orbits $G\cdot\binom{u}{v}$:
  (i) $(0,0)$ is an orbit.
  (ii) $\{u\} \times \mathbb{R}$, where $(u,v) \neq (0,0)$
  (iii) $\{0\} \times (\mathbb{R}\setminus\{0\})$ where $\left(\begin{smallmatrix} u = 0 \\ v \neq 0 \end{smallmatrix}\right)$.

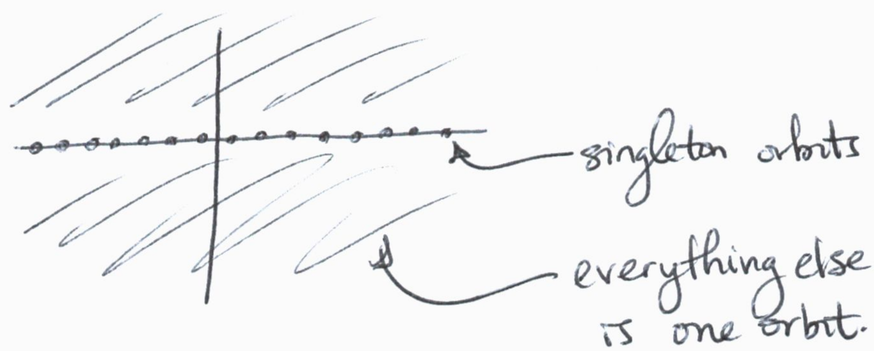
i.e. $\mathfrak{g} =$



a decomposition into orbits.

- coadjoint orbits:
  (i) $G\cdot[\zeta, 0] = \{[\zeta, 0]\}$
  (ii) $\underset{\eta \neq 0}{G\cdot[\zeta, \eta]} = \mathbb{R}^2 \setminus \{\zeta\text{-axis}\}$.

Then $\mathfrak{g}^* =$



<u>Remark</u>: Something true in general is that coadjoint orbits are always <u>even dimensional</u>!

Example 2: Let $G = SO(3) = \{ A \mid A^TA = 1, \det(A) = 1 \}$.

Then $\mathfrak{g} = \{ a \mid a^T + a = 0, \ tr(a) = 0 \}$. We want to identify each with a copy of $\mathbb{R}^n$. Matrices with $a^T + a = 0$ have zeroes on the diagonal, so they in general are

$$\begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & a_1 \\ a_2 & a_1 & 0 \end{pmatrix} = \hat{a} \longleftarrow\!\!\!\!\!+ \ a \in \mathbb{R}^3.$$

A direct calculation would be:

$$\hat{a} b = a \times b, \quad \text{moreover } [\hat{a}, \hat{b}] = 2\widehat{a \times b}$$

This says we have a Lie algebra isomorphism, with the bracket operation in the matrix group corresponding to the cross product of vectors.

Question: Is there a nice geometric reason in $\mathbb{R}^3$ for the Jacobi identity to hold?

We can check:

$$[\hat{a}, \hat{b}]c = (\hat{a}\hat{b} - \hat{b}\hat{a})c = \hat{a}\hat{b}c - \hat{b}\hat{a}c$$
$$= 2(a \times b) \times c = 2(\widehat{a \times b})c.$$

So the action on vectors c is the same, i.e.

$$[\hat{a}, \hat{b}] = 2(\widehat{a \times b}).$$

The adjoint representation is:

$$Ad_A \hat{a} = A \hat{a} A^{-1}, \text{ so}$$

$$(Ad_A \hat{a})c = A\hat{a}A^{-1}c = A(a \times A^{-1}c) \quad \text{use } Ma \times Mb = \det M \cdot M^T a \times b$$
$$= Aa \times c$$

$= (\hat{Aa})c.$

Therefore $Ad_A \hat{a} = A\hat{a}$ ie the adjoint representation action is $SO(3)$ acting on $\mathbb{R}^3$ by rotations.

- The coadjoint representation is also rotation.
- The orbits of a vector under rotation are the origin and nested spheres centered at the origin.

Note: Again, even dimensional orbits.

Poisson structures.

Def: A Poisson structure on a manifold M is

$$\{\ ,\ \}: C^\infty(M) \times C^\infty(M) \longrightarrow C^\infty(M)$$

such that $\{\ ,\ \}$ gives $C^\infty(M)$ a Lie algebra structure, and $\{\ , h\}$ is a derivation. I.e:

$$\{fg, h\} = \{f, h\}g + f\{g, h\}.$$

Ex: • $\{\ ,\ \} = 0$ is a Poisson structure, ie there's no topological obstruction to a manifold M admitting a Poisson structure.

- $M = \mathbb{R}^2$, then

$$\{f, g\} = \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}$$

- $\mathfrak{g}^*$ always has a Poisson structure.

Given a function $f : \mathfrak{g}^* \longrightarrow \mathbb{R}$, then

$$df|_0 : \mathfrak{g}^* \longrightarrow \mathbb{R} \text{ (linear map)}$$

Then $df|_0 \in (\mathfrak{g}^*)^* = \mathfrak{g}$ and we have

$\{f,g\}(\mu) = \langle \mu, [df, dg] \rangle$; the Kirillov-Kostant-Souriau Poisson structure.

Every Poisson manifold hase a decomposition into even dimensional symplectic immersed submanifolds.