Intro to Riemann Surfaces.

Why is complex analysis natural? Take an "unavoidable" equation, like $\Delta u = 0$.
Suppose we try to factor the operator, say

$$\left(\frac{\partial}{\partial x} + \alpha \frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} + \beta \frac{\partial}{\partial y}\right) u = 0$$

$$\parallel$$

$$\Delta u.$$

Then you need $\alpha\beta = 1$, $\alpha = -\beta$. Then real numbers do not work, complex numbers are forced upon you; alternatively try α, β matrices. Then a solution would be $\alpha = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\beta = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Then our solutions come in pairs $\begin{pmatrix} u \\ v \end{pmatrix}$ instead, we could try to find solutions of the form

$$\left(\frac{\partial}{\partial x} + \beta \frac{\partial}{\partial y}\right)\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff \begin{pmatrix} u_x - v_y \\ u_y - v_x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left(\frac{\partial}{\partial x} + \alpha \frac{\partial}{\partial y}\right)\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff \begin{pmatrix} u_x + v_y \\ v_x - u_y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

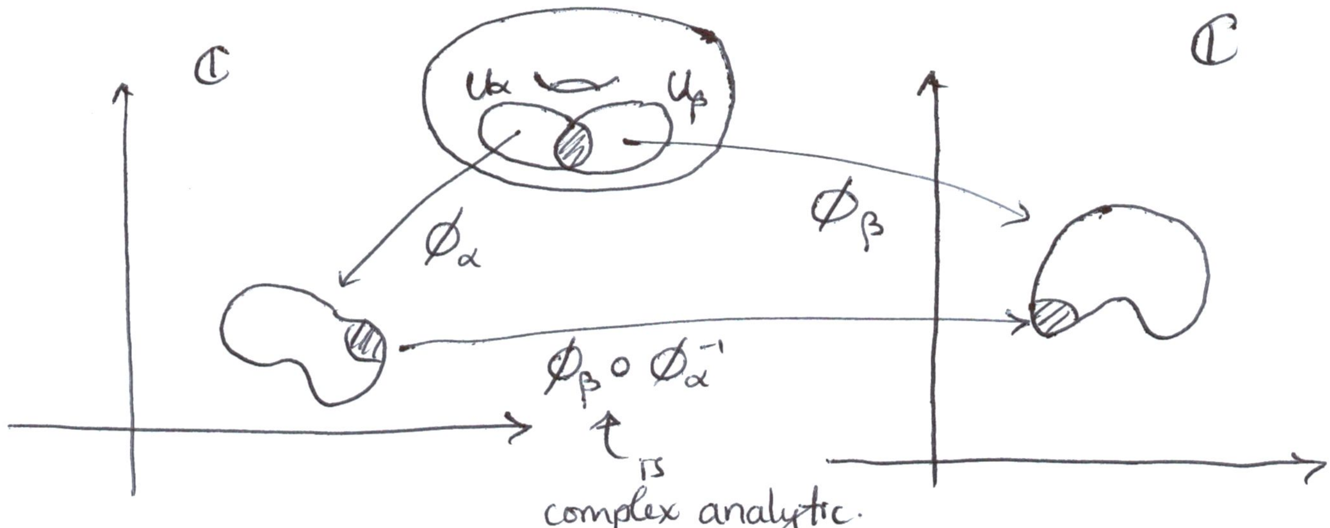
Def of a Riemann surface

It is a Hausdorff second countable topological space R s.t. (i) $\forall x \in R$ $\exists$ a homeomorphism

$\phi : \mathcal{U} \longrightarrow V \subseteq \mathbb{C}$ ($\mathcal{U}, V$ open) with $x \in \mathcal{U}$.

(ii) there is collection $\{\phi_\alpha, \mathcal{U}_\alpha\}$ of these charts such that $\phi_\beta \circ \phi_\alpha^{-1}\Big|_{\phi_\alpha(\mathcal{U}_\alpha \cap \mathcal{U}_\beta)}$ is a complex analytic homeomorphism onto its image.

(iii) The charts cover R.



Examples: • Any open subset of $\mathbb{C}$

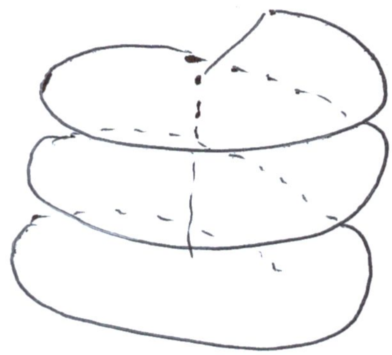
• The Riemann sphere $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

• $R = \{(r, \theta) \mid r > 0 \text{ and } \theta \in \mathbb{R}\}$. We take $\mathcal{U}_{s,a,b} = (0, s) \times (a, b)$, with $b - a < \pi$, then $\phi_{s,a,b}(r, \theta) = re^{i\theta}$. Then the <u>transition functions</u> $\phi_{s_1,a_1,b_1} \circ \phi_{s_2,a_2,b_2}^{-1}$ are complex analytic, and if you have time they work out to be quite nice.

This last example is a bit easier to think of in this way: (but we abuse notation here)

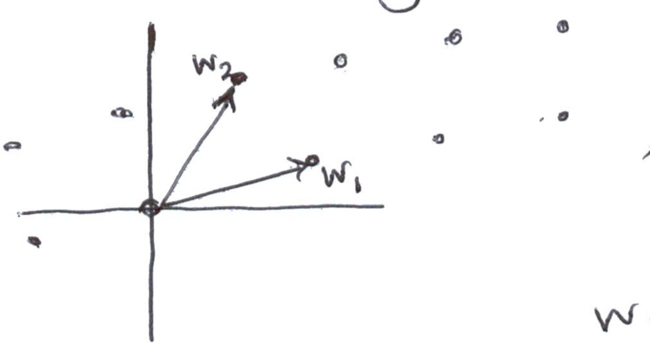
$$\tilde{R} = \{ re^{i\theta} \mid r > 0,\ \theta \in \mathbb{R} \}$$ where

we think of $re^{i\theta}$ and $re^{i(\theta+2\pi)}$ as distinct points.



you get a sort of spiral.

- Γ a lattice, $\Gamma = \mathrm{span}_{\mathbb{Z}}\{w_1, w_2\}$ where w_1, w_2 are $\mathbb{R}$-linearly independent.



, $R = \mathbb{C}/\Gamma$ so

$w \sim z \Leftrightarrow w - z = nw_1 + mw_2$,

equipped with the quotient topology.

Then we have fundamental domains

and charts are $(\phi, \mathcal{U})$ and $\tilde{\mathcal{U}} \subseteq \mathbb{C}$ is a small enough open set so that it doesn't contain "duplicates", ie. multiple elements of an equivalence class. Then

$$\mathcal{U} = \pi(\tilde{\mathcal{U}}) \quad \text{where} \quad \pi : \mathbb{C} \longrightarrow \mathbb{C}/\Gamma,$$

$\phi = \left(\pi\big|_{\tilde{\mathcal{U}}}\right)^{-1}$, and transition functions are

translations by $nw_1 + mw_2$.

Back to our original motivation of $\Delta u = f$, solving PDE's. We need $\Delta u = \delta(0)$.
This means $\iint u \Delta v = v(0)$ $\forall v$ s.t. v is C^∞, and compactly supported.

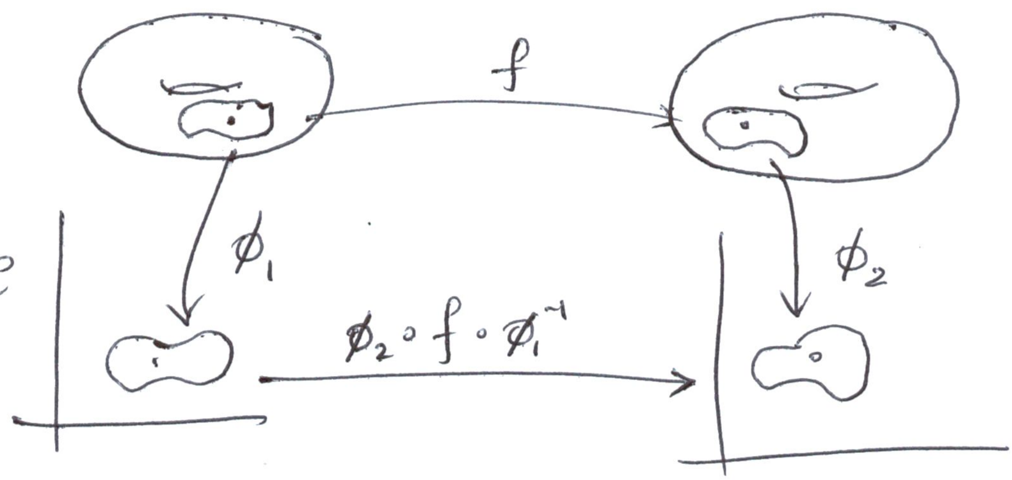
Answer: $u = \log|z|$, up to a constant factor.
What's the conjugate?

$$\tilde{u}(z) = \int_{z_0}^{z} \tilde{u}_x \, dx + \tilde{u}_y \, dy$$

$$= \int_{z_0}^{z} -u_y \, dx + u_x \, dy = \arg z, \text{ and}$$

e.g.
$f(z) = \log z = \log|z| + i\arg z$ is only well-defined on the Riemann surface R from two examples ago
$(e^{i\theta} \neq e^{i(\theta+2\pi)})$

Let R_1 and R_2 be Riemann surfaces. We say $f: R_1 \longrightarrow R_2$ is complex analytic if $\forall$ charts ϕ_1 on R_1 and ϕ_2 on R_2, $\phi_2 \circ f \circ \phi_1^{-1}$ is complex analytic: Eg.

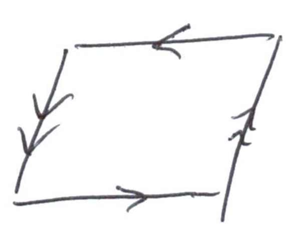


<u>exercise</u>: This is consistent, ie. independent of choice of charts.

<u>Example</u>: $f: \mathbb{C}/\Gamma \longrightarrow \mathbb{C} \cup \{\infty\}$ say f has at worst poles, otherwise complex analytic.

- If f is not $\equiv 0$, then the number of 0's is equal to the number of poles (with multiplicity).

To see this, take γ going once around a fundamental domain (avoiding zeroes and poles).



Then # zeros − # poles

$$= \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} dz$$

$$= 0.$$

One can also show $\sum_i \mathrm{Res}(f) = 0$.

Which tori are equivalent (as Riemann surfaces)?

We sweep under the rug: All Riemann surfaces homeomorphic to a torus are complex analytic equivalent to a torus of the form $\mathbb{C}/\Gamma$. Then we can restrict to asking about tori of the form $\mathbb{C}/\Gamma$, and the question becomes:

When are two lattices equivalent?

$$\mathbb{C}/\Gamma \sim \mathbb{C}/\Gamma' \iff \exists\, \mu \in \mathbb{C}\setminus\{0\} \text{ s.t. } \mu\Gamma = \Gamma'.$$

Second thing we must take for granted:

Given $f: \mathbb{C}/\Gamma \to \mathbb{C}/\Gamma'$, $\exists\, g: \mathbb{C} \to \mathbb{C}$ s.t. the following commutes

$$\begin{tikzcd} \mathbb{C} \arrow[r, "g"] \arrow[d, "\pi"'] & \mathbb{C} \arrow[d, "\pi"] \\ \mathbb{C}/\Gamma \arrow[r, "f"] & \mathbb{C}/\Gamma' \end{tikzcd}$$

, where g is a biholomorphism.

So $g(z) = \mu z + \eta$.

For g to descend to the quotient, we need

$$\mu z_1 + \eta \cong \mu z_2 + \eta \quad \text{whenever } z_1 - z_2 \in \Gamma$$

need $\eta\Gamma \subseteq \Gamma'$ and $\eta\Gamma \supseteq \Gamma'$.

So we still need to enumerate the lattices somehow, even if we now have a notion of equivalence.

When are $\Gamma = \langle w_1, w_2 \rangle = \mathrm{span}_{\mathbb{Z}}\{w_1, w_2\}$

and $\Gamma' = \langle w_1', w_2' \rangle = \mathrm{span}_{\mathbb{Z}}\{w_1', w_2'\}$

the same lattice?

<u>Answer</u>: $\langle w_1, w_2 \rangle = \langle w_1', w_2' \rangle$

$\Longleftrightarrow \exists\, M \in M_{2\times 2}(\mathbb{Z})$ s.t. $\det M = \pm 1$ and

$$\begin{pmatrix} w_2' \\ w_1' \end{pmatrix} = M \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}.$$

Then we can always assume $w_1 = 1$ and $w_2 = \tau$ where $\tau \in \mathbb{H} = \{z \mid \mathrm{Im}(z) > 0\}$. (We can assume this since $(w_1, w_2) \sim \left(1, \frac{w_2}{w_1}\right)$, possibly up to reordering the basis).

<u>Theorem</u>: $\langle 1, \tau \rangle \sim \langle 1, \tau' \rangle$ (i.e. $\mathbb{C}/\langle 1,\tau\rangle \sim \mathbb{C}/\langle 1, \tau' \rangle$)

$\Longleftrightarrow \tau = T(\tau')$ for some $T(w) = \dfrac{aw+b}{cw+d}$, where

$a, b, c, d \in \mathbb{Z}$ and $ad - bc = 1$.

<u>Proof</u>: $\mathbb{C}/\langle 1,\tau\rangle \sim \mathbb{C}/\langle 1, \tau' \rangle \Longleftrightarrow \langle 1, \tau \rangle = \langle \nu 1, \nu \tau \rangle$
some $\nu \in \mathbb{C} \setminus \{0\}$

$$\Rightarrow \begin{pmatrix} \tau' \\ 1 \end{pmatrix} = \begin{pmatrix} \mu a \tau + \mu b \\ \mu c \tau + \mu d \end{pmatrix},$$ $\boxed{\text{assuming } a,b,c,d \in \mathbb{Z},\ ad-bc = 1 \text{ then}}$ (*).

$\tau' = \dfrac{a\tau + b}{c\tau + d}$ with (*). Reverse steps by setting $\mu = \dfrac{1}{c\tau + d}$.