Fuchsian groups and the uniformization theorem.

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1. Riemann surfaces.

A Riemann surface is a two-dimensional topological manifold. (Hausdorff, second countable). Moreover, there is an atlas $\mathcal{A} = \{(U_\alpha, \phi_\alpha)\}$ satisfying

(i) U_α is open, $\phi_\alpha : U_\alpha \longrightarrow V \subseteq \mathbb{R}^2$ is a homeomorphism.

(ii) $\bigcup_\alpha U_\alpha = R$, the whole Riemann surface.

(iii) Whenever $U_\alpha \cap U_\beta \neq \emptyset$, $\phi_\alpha \circ \phi_\beta^{-1}$ is 1-1 holomorphic.

Examples:

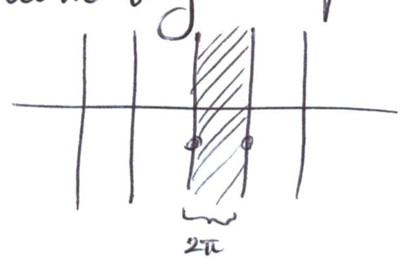
- $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, using as a topology the typical open sets, together with sets $\{z \mid |z| > r\} \cup \{\infty\}$ to generate it.

  An atlas is $\{(\mathbb{C}, z \mapsto z), (\overline{\mathbb{C}} \backslash \{0\}, z \longmapsto \frac{1}{z})\}$.

- $G = \{g_n(z) \mid g_n(z) = z + n\tau, n \in \mathbb{Z}\}$ (here $\tau \in \mathbb{C} \backslash \{0\}$)

Then $\mathbb{C}/G$ is a Riemann surface: $z \sim w$ if $\exists g_n$ s.t. $g_n(z) = w$.

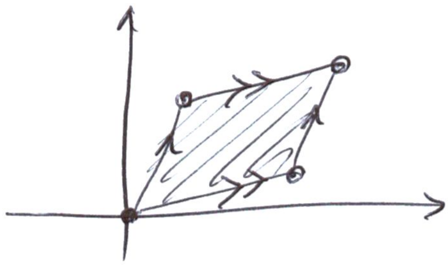
E.g. If $\tau = 2\pi$, then we're quotienting the plane by horizontal translation by 2π:



and the quotient space is an infinite cylinder.

Note $\exp : \mathbb{C}/G \longrightarrow \mathbb{C}\setminus\{0\}$ is a well defined
$z \longmapsto e^{iz}$ 1-1 holomorphic, ie. conformal, map.

- $G = \{ g_n(z) = z + n\tau_1 + m\tau_2 \mid n, m, \in \mathbb{Z} \}$. Here τ_1, τ_2 are fixed $\mathbb{R}$-linearly independent vectors. Then we get



and $\mathbb{C}/G$ is topologically a torus.

<u>Definition</u>: A fundamental domain of G is a set D s.t.

(i) $\bigcup_{g \in G} g(D) = \mathbb{C}$

(ii) $g(D^\circ) \cap h(D^\circ) \neq \emptyset \quad \forall g, h \in G$.

<u>Definition</u>: A <u>holomorphic</u> <u>covering</u> is a map $\pi : R \longrightarrow R_1$ where R, R_1 are Riemann surfaces such that

(1) π is holomorphic

(2) $\forall w \in R_1$ and fixed $z = \pi^{-1}(w) \in R$, $\exists$ open $U \ni w$ and $V \ni z$ such that $\pi|_V$ is a conformal map onto U.

We say R is <u>the</u> universal cover if R is simply connected.

Deck transformations of the universal cover are $g: R \to R$ (homeomorphism) s.t. $\pi \circ g = \pi$. Note that g is conformal, since locally $g = \pi^{-1} \circ \pi$.

<u>Theorem</u>: (Uniformization)

(a) Every simply connected Riemann surface is conformal to either $\mathbb{C}$, $\overline{\mathbb{C}}$ or $\mathbb{D} = \{z \mid |z| < 1\}$

(b) Every Riemann surface is covered by either $\mathbb{C}$, $\overline{\mathbb{C}}$ or $\mathbb{D}$.

<u>Question</u>: What's covered by $\overline{\mathbb{C}}$?

<u>Ans</u>: Just $\overline{\mathbb{C}}$.

<u>Question</u>: What's covered by $\mathbb{C}$?

<u>Ans</u>: Up to biholomorphism, only example 2 and example 3.

<u>Question</u>: What's covered by $\mathbb{D}$?

<u>Ans</u>: Everything else.

So, to study Riemann surfaces means to study spheres, cylinders, tori or things covered by the disk.

<u>Part 2</u>: Fuchsian groups.

<u>Definition</u>: A Fuchsian group G is a set of conformal maps $\mathbb{D} \to \mathbb{D}$ s.t. $\forall z \in \mathbb{D}$ $\exists\, U$ open s.t. $g(U) \cap U = \emptyset$ $\forall g \neq id$ in G.

Note: Sometimes we allow $g(U) \cap U \neq \emptyset$ for finitely many $g \in G$. This allows for branched covers.

Theorem: Except for $\overline{\mathbb{C}}$, $\mathbb{C}$, $\mathbb{C} \setminus \{0\}$, tori, any Riemann surface is given by $\mathbb{C}/G$ for G Fuchsian.

Theorem: $T : \mathbb{D} \to \mathbb{D}$ is conformal $\Longleftrightarrow T(z) = e^{i\theta} \dfrac{z-a}{1-\bar{a}z}$ $a \in \mathbb{D}$.

Call maps of this form automorphisms, $\mathrm{Aut}(\mathbb{D})$.

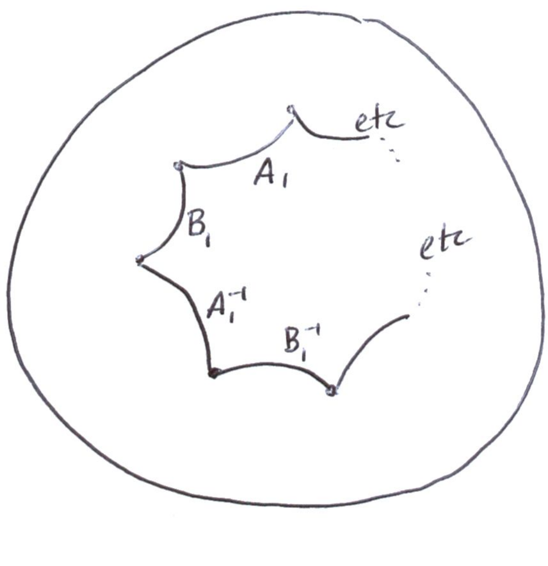
Definition: The hyperbolic distance between $z, w \in \mathbb{D}$ is

$$d(z,w) = \frac{1}{2} \log \left(\frac{1 + \left| \frac{z-w}{1-\bar{w}z} \right|}{1 - \left| \frac{z-w}{1-\bar{w}z} \right|} \right) = \operatorname{arctanh}\left(\left| \frac{z-w}{1-\bar{w}z} \right| \right).$$

Theorem: $\mathrm{Aut}(\mathbb{D}) = \mathrm{Isom}(\mathbb{D}, d)$.

Theorem: The shortest path between two points $z, w \in \mathbb{D}$ is a circle intersecting $\partial \mathbb{D}$ at right angles.

Example: (Poincaré)



Take geodesics labelled as left, forming a $4n$-gon.

We need

(i) angles add to 2π

and (ii) length (A_i) = length (A_i^{-1})

length (B_i) = length (B_i^{-1}).

~~AA~~ Now there is a unique $\alpha_i \in Aut(\mathbb{D})$ s.t. $\alpha_i(A_i) = A_i^{-1}$

and β_i s.t. $\beta_i(B_i) = B_i^{-1}$.

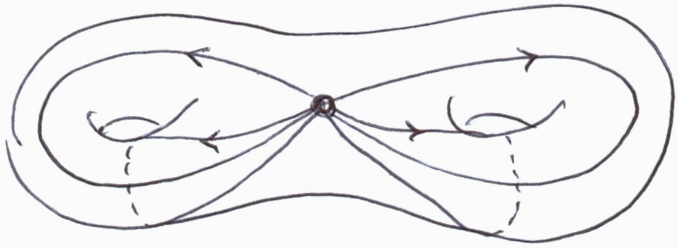
Then let $G = \langle \alpha_i, \beta_i ; i = 1, \dots, n \rangle \subset Aut(\mathbb{D})$ be the group they generate. Then

(i) The polygon ~~generated~~ bounded by A_i's and B_i's is a fundamental domain

(ii) G is Fuchsian

(iii) $\mathbb{D}/G$ is a compact surface of genus n.

and



(cut and get octagon.)

Example: $\mathbb{H} = \{ z \mid Im(z) > 0 \}$

$G = \{ g_n(z) = r^n z \mid n \in \mathbb{Z} \}$, $r > 1$.

Then a fundamental domain is