MATH 1500 February 10

Lecture 16.

Recall last day we learned the chain rule:

$f(g(x)) = f'(g(x)) \cdot g'(x)$ when $f'(x)$ and $g'(x)$ both exist.

Or, if $y=f(u)$ and $u=g(x)$, we can write

$$\frac{dy}{dx} = \frac{df}{du} \cdot \frac{du}{dx}$$

Example: Suppose that $y = \sin(\tan(2x))$. What is y'?

Solution: Here, we can build y out of 3 functions

$f(u) = \sin(u)$, $u(v) = \tan(v)$ and $v(x) = 2x$.

Then $y = f(u(v(x))) = f(u(2x)) = f(\tan(2x))$
$= \sin(\tan(2x))$.

Then the Leibniz form of the chain rule gives

$\frac{dy}{dx} = \frac{df}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}$, a "triple chain rule"

since we have a composition of 3 functions. We calculate:

$$\frac{df}{du} = \frac{d}{du}(\sin(u)) = \cos(u)$$

$$\frac{du}{dv} = \frac{d}{dv}(\tan(v)) = (\sec(v))^2$$

$$\frac{dv}{dx} = \frac{d}{dx}(2x) = 2.$$

So the derivative $\frac{dy}{dx}$ is

$$\frac{dy}{dx} = \cos(u)\cdot(\sec(v))^2\cdot 2$$

But we want our final answer in terms of x, not u and v. So we sub in $v = 2x$ and $u = \tan(v) = \tan(2x)$.

$$\frac{dy}{dx} = \cos(\tan(2x))\,(\sec(2x))^2\cdot 2.$$

§3.5. Questions 5-20, 25-32.

Sometimes we are given an equation where we cannot solve for y, but we still want to compute $\frac{dy}{dx}$. It is possible, using implicit differentiation.

Trick: If you are given an equation where you cannot solve for y, think of y as a function $y(x)$ and differentiate it using the chain rule!

Ie. If y^3 appears in an equation, then we will write $\frac{d}{dx}(y^3) = \frac{d}{dy}(y^3)\cdot\frac{dy}{dx} = 3y^2\frac{dy}{dx}$ (chain rule).

Example: If $x^3 + y^3 = 6xy$, what is $\frac{dy}{dx}$?

Solution: We take derivatives $\frac{d}{dx}$ of both sides

$$\frac{d}{dx}(x^3+y^3) = \frac{d}{dx}(6xy).$$

The left side is

$$\frac{d}{dx}(x^3) + \frac{d}{dx}(y^3)$$

$$= 3x^2 + 3y^2 \frac{dy}{dx} \quad \text{(Chain rule used on } y = y(x)\text{).}$$

The right hand side is a product, the product rule gives

$$\frac{d}{dx}(6xy) = 6\frac{d}{dx}(xy) = 6\left[y\frac{d}{dx}(x) + x\frac{dy}{dx}\right]$$

$$= 6y + 6x\frac{dy}{dx}.$$

So overall we have

$$3x^2 + 3y^2\frac{dy}{dx} = 6y + 6x\frac{dy}{dx}$$

and we can solve for $\frac{dy}{dx}$!

$$3x^2 - 6y = 6x\frac{dy}{dx} - 3y^2\frac{dy}{dx} = (6x - 3y^2)\frac{dy}{dx}$$

So $\frac{dy}{dx} = \frac{3x^2 - 6y}{6x - 3y^2} = \frac{x^2 - 2y}{2x - y^2}$.

<u>Example</u>:

Find y' if $x^2 + 3y^2 = 4$. Find the equation of the tangent line at $(1,1)$.

<u>Solution</u>: The chain rule applied to $3y^2$ (remembering $y = y(x)$) gives

$$(3y^2)' = 3 \cdot 2y \cdot y' = 6yy'.$$

So taking derivatives of both sides gives

$$(x^2 + 3y^2)' = (4)'$$

$$2x + 6yy' = 0.$$

Therefore solving for y' gives $y' = \frac{-2x}{6y}$.

So, the slope of the tangent line at the point $(1,1)$ is $y' = \frac{-2(1)}{6(1)} = \frac{-1}{3}$. Therefore the equation is

$y = \frac{-1}{3}x + b$, with b chosen so the line passes through $(1,1)$.

Therefore $1 = \frac{-1}{3}(1) + b \Rightarrow b = 1 + \frac{1}{3} = \frac{4}{3}$.

So the tangent line is $y = \frac{-1}{3}x + \frac{4}{3}$.

<u>Example</u>: Find $\frac{dy}{dx}$ if $e^{y^2} = yx$.

<u>Solution</u>: The chain rule applied to the left hand side is complicated. If $g(u) = e^u$ and $u(y) = y^2$, then $g(u(y)) = e^{y^2}$. So we get

$$\frac{d}{dx}(e^{y^2}) = \frac{dg}{du} \cdot \frac{du}{dy} \cdot \frac{dy}{dx}$$

$$= e^u \cdot 2y \cdot \frac{dy}{dx} = e^{y^2} \cdot 2y \frac{dy}{dx}$$

The right hand side is a product:

$$\frac{d}{dx}(yx) = y\cancel{\frac{dx}{dx}} + x\frac{dy}{dx} = y + x\frac{dy}{dx}$$

So we have

$$2ye^{y^2}\frac{dy}{dx} = y + x\frac{dy}{dx}$$

Solving... $2ye^{y^2}\frac{dy}{dx} - x\frac{dy}{dx} = y$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{2ye^{y^2}-x}$$

<u>Example</u>: Find $\frac{dy}{dx}$ if $x = \sin y$ and $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$.

<u>Note</u>: We ask for $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$ to narrow down the possibilites for y, otherwise $x = \sin y$ has many solutions for each x-value

and we cannot think of $y = y(x)$ as a function of x.

Then we use implicit differentiation:

$$\frac{d}{dx}(x) = \frac{d}{dx}(\sin y) = \cos(y)\frac{dy}{dx}$$

So $1 = \cos(y)\frac{dy}{dx}$, or $\frac{dy}{dx} = \frac{1}{\cos(y)}$.

We can actually write this in terms of x, since $(\cos(y))^2 + (\sin(y))^2 = 1$

$$\Rightarrow \cos(y) = \sqrt{1-(\sin(y))^2} = \sqrt{1-x^2}$$

so $$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

In fact, this formula is the first inverse trig derivative formula.

Recall that $\quad \sin^{-1}(x) = y$ means $\sin(y) = x$ and $-\pi/2 \le y \le \pi/2$.

So the calculation of $\frac{dy}{dx}$ that we just did is

$$\frac{dy}{dx} = \frac{d}{dx}\left(\sin^{-1}(x)\right) = \frac{1}{\sqrt{1-x^2}}.$$

In general we have a bunch of new formulas:

$$\frac{d}{dx}\left(\cos^{-1}(x)\right) = -\frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\left(\csc^{-1}(x)\right) = -\frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}\left(\tan^{-1}(x)\right) = \frac{1}{1+x^2} \qquad \frac{d}{dx}\left(\sec^{-1}(x)\right) = \frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}\left(\cot^{-1}(x)\right) = \frac{-1}{1+x^2}.$$

YIKES.

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§3.5 finished

Last day we saw implicit differentiation, a way of finding $\frac{dy}{dx}$ when we cannot solve for y.

Example: Find y'' if $x^3 + 4y^3 = 5$.

Solution. Recall we think of $y = y(x)$ as a function of x, and then use the chain rule on $4y^3$. We get, upon taking derivatives of both sides:

$$(x^3 + 4y^3)' = (5)'$$

$$\Rightarrow 3x^2 + 4\cdot(3y^2y') = 0.$$

$$\Rightarrow 3x^2 + 12y^2y' = 0.$$

Now we can solve for y' and get... something with y's again. No help. So we have to keep differentiating implicitly.

$$(3x^2 + 12y^2y')' = (0)'$$

$$\Rightarrow 6x + 12(y^2y')' = 0.$$

We use the product rule on $(y^2y')'$ and get

$$(y^2y')' = (y^2)'y' + y''y^2 = 2y(y')^2 + y''y^2.$$

So overall,

$$6x + 12(2y(y')^2 + y''y^2) = 0.$$

So we rearrange:

$$6x + 24y(y')^2 + 12y''y^2 = 0$$

$$\Rightarrow \quad y'' = \frac{-6x - 24y(y')^2}{12y^2} = \frac{-x - 4y(y')^2}{2y^2}.$$

§3.9 Related rates. Do all problems not requiring a calculator or graphing calculator, but at least do 1-14, 20-24.

Relate rates is our first real-world application. The idea is you have:

1. Information given about the rate of change of one thing, and you are asked something about the rate of change of another thing.
2. You need to find an equation that relates the two changing variables
3. Then differentiate the equation from part 2 in order to find an equation between the two rates of change in 1.

Example: A balloon is perfectly spherical and being filled with 100cm^3 of gas per second. When the diameter of the balloon is 50cm, how fast is its radius changing?

Solution: $\boxed{\text{Part ①}}$ We identify and name the two changing quantities.

Quantity 1: Volume, we will write $V(t)$ since it is a number changing over time.

Quantity 2: Radius of the balloon, we will write $r(t)$ since it is a number r changing over time.

Given: $\frac{dV}{dt}$ = rate of change of volume = $100 \text{cm}^3/\text{sec}$

Want to find: $\frac{dr}{dt}$ when diameter = 50 cm, ie. $r(t) = 25$ cm.

$\boxed{\text{Part ②}}$ Write an equation relating the quantities $V(t)$ = volume of sphere and $r(t)$ = radius of sphere

The volume formula gives $V(t) = \frac{4}{3}\pi(r(t))^3$.

ie. $V = \frac{4}{3}\pi r^3$.

$\boxed{\text{Part ③}}$ Differentiate the equation from ② and use the given data from ①. We want $\frac{d}{dt}$ so we differentiate:

$$\frac{d}{dt}(V(t)) = \frac{d}{dt}\left(\frac{4}{3}\pi(r(t))^3\right)$$

We use the chain rule on the right hand side:

$$\frac{dV}{dt} = \frac{4}{3}\pi\left(3(r(t))^2\frac{dr}{dt}\right)$$

$$= 4\pi(r(t))^2\frac{dr}{dt}.$$

The given quantities are plugged in: ($\frac{dV}{dt} = 100$ and $r(t) = 25$)

$$100 = 4\pi(25)^2 \frac{dr}{dt}, \text{ so } \frac{dr}{dt} = \frac{100}{4\pi(25)^2} = \frac{1}{25\pi}.$$

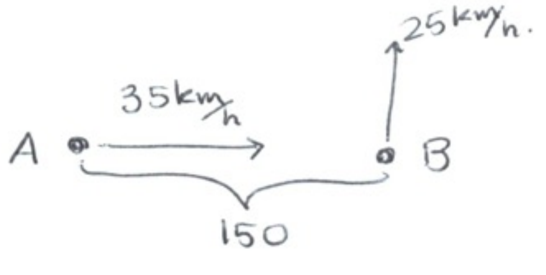
So, when $r(t) = 25$ and we're pumping in $100\,cm^3/s$, the radius is changing at $\frac{1}{25\pi}$ cm/s.

<u>Note</u>: In the book they offer a 7-step breakdown instead of 3.

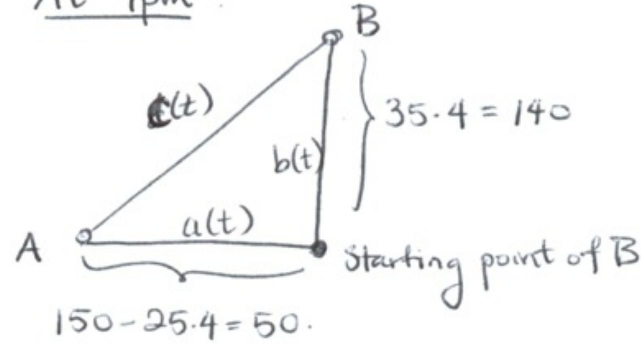
(14) At noon, ship A is 150km west of ship B. Ship A is sailing east at 35 km/h and B is sailing north at 25km/h. How fast is the distance between the ships changing at 4:00pm?

<u>Solution</u>: Draw a picture when you can!

<u>At noon</u>:



<u>At 4pm</u>:



$\boxed{\text{Step 1}}$

The quantities we know are:

- $a(t)$ is the distance from ship A to ship B's starting point, and $a(4) = 50$, $\frac{da}{dt} = -35$
- $b(t)$ is the distance from ship B to its starting point, $b(4) = 140$ and $\frac{db}{dt} = 35$

- $c(t)$ is the distance between ship A and ship B. We know $c(4) = \sqrt{(a(4))^2 + (b(4))^2}$

$$= \sqrt{2500 + 19600}$$

<u>We want</u> $\left.\frac{dc}{dt}\right|_4$. $\simeq 148.66$.

$\boxed{\text{Step 2}}$ An equation relating all the knowns and unknowns is Pythagorean theorem:

$a^2 + b^2 = c^2$ or $(a(t))^2 + (b(t))^2 = (c(t))^2$.

We differentiate with respect to t:

$$\frac{d}{dt}(a(t))^2 + \frac{d}{dt}(b(t))^2 = \frac{d}{dt}(c(t))^2$$

So $2a(t)\frac{da}{dt} + 2b(t)\frac{db}{dt} = 2c(t)\frac{dc}{dt}$

$\boxed{\text{Step 3}}$ Plug in unknowns and solve for unknown rate:

$$2a(4)\left.\frac{da}{dt}\right|_4 + 2b(4)\left.\frac{db}{dt}\right|_4 = 2c(4)\left.\frac{dc}{dt}\right|_4$$

$$2(50)(-35) + 2(140)(35) = 2(148.66)\left.\frac{dc}{dt}\right|_4.$$

<u>Solving</u>, $\left.\frac{dc}{dt}\right|_4 = \frac{6300}{297.3} \simeq 21.2$. So the ships are moving apart at 21.2 km/h.

(5) A cylindrical tank with radius 5m is being filled with water at $3 m^3/min$. How fast is the depth in the tank increasing?

Solution: $\boxed{\text{Step 1}}$ Identify known info and draw a picture

We know that $V(t)$ is the volume, and $\frac{dV}{dt} = 3$.

$h(t)$, height. We want to know $\frac{dh}{dt}$.

$\boxed{\text{Step 2}}$ Relate knowns and unknowns by an equation

The volume of the water in the tank is

$$V(t) = \pi r^2 h(t)$$
$$= 25\pi h(t).$$

$\boxed{\text{Step 3}}$ Differentiate with respect to t, plug in knowns and solve.

$$\frac{d}{dt}(V(t)) = \frac{d}{dt} 25\pi (h(t))$$
$$\Rightarrow \frac{dV}{dt} = 25\pi \frac{dh}{dt}.$$

So plugging in $\frac{dV}{dt} = 3$, we get $3 = 25\pi \frac{dh}{dt}$

or $\frac{dh}{dt} = \frac{3}{25}\pi$.

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Related rates continued...

Example: Gravel is dumped from a conveyor belt at a rate of $30 m^3/min$, forming a pile in the shape of a cone whose base diameter and height are always equal. How fast is the height of the pile increasing when it is 10m high?

Solution: $\boxed{\text{Step 1}}$ Picture:

List all quantities, knowns and unknowns.

- The height of the pile, $h(t)$
- The base diameter of the pile, $d(t)$.
- We're told $h(t) = d(t)$
- The volume $V(t)$ of the pile, we're told $\frac{dV}{dt} = 30\, m^3/min$.
- Asked to find $\frac{dh}{dt}$, assuming $h(t) = d(t) = 10$.

$\boxed{\text{Step 2}}$. Find an equation relating the quantities involved. The volume of a cone is

$$V = \frac{1}{3}\pi r^2 h.$$

So we replace r with $\frac{1}{2}d(t)$, and get

$$V(t) = \frac{1}{3}\pi\left(\frac{1}{2}d(t)\right)^2 h(t)$$
$$= \frac{1}{3}\pi \cdot \frac{1}{4}(d(t))^2 \cdot h(t)$$

We can use $d(t) = h(t)$ to simplify:

$$V(t) = \frac{1}{12}\cdot\pi\cdot(h(t))^3$$

$\boxed{\text{Step 3}}$ Implicitly differentiate all quantities and solve for the unknown.

$$\frac{d}{dt}\left(V(t)\right) = \frac{d}{dt}\left(\frac{\pi}{12}(h(t))^3\right)$$
$$\Rightarrow \frac{dV}{dt} = \frac{\pi}{12}3(h(t))^2\frac{dh}{dt}$$

So we plug in $\frac{dV}{dt} = 30$, $h(t) = 10$ and get

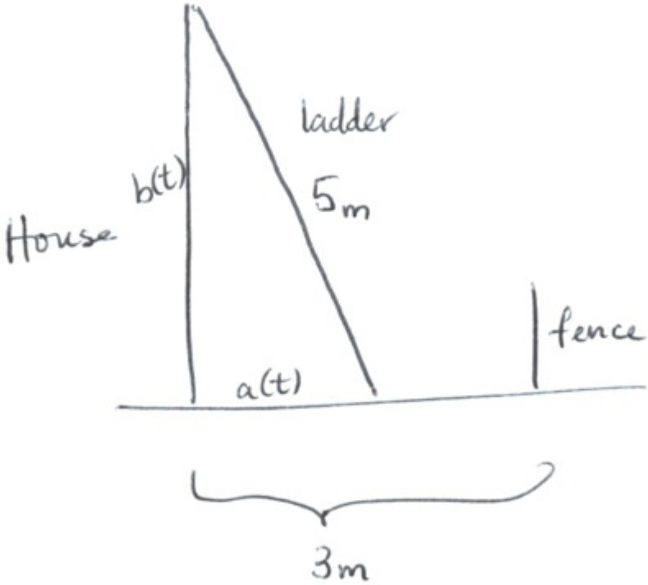
$$30 = \frac{\pi}{12}3(10)^2\frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{30\cdot 12}{300\cdot\pi} = \frac{6}{5\pi}\,m/min.$$

or $\simeq 0.38\,m/min.$

Example: A house is 3m away from a fence marking the edge of the property. A ladder 5m long is leaning against the house, and the top begins to slide down the house at 1m/s.
A what speed does the end of the ladder strike the fence?

Solution: $\boxed{\text{Step 1}}$ Picture and list all quantities, known and unknown.

Picture



- $a(t)$ is distance from the foot of the ladder to the house. Want $\frac{da}{dt}$ when $a(t) = 3$.

- $b(t)$ is height of the ladder, $\frac{db}{dt} = -1$ and when $a(t) = 3$, $b(t)$ is found using

$$(a(t))^2 + (b(t))^2 = 5^2$$
$$\Rightarrow (b(t))^2 = 5^2 - 3^2$$
$$\Rightarrow b(t) = \sqrt{25-9} = 4.$$

$\boxed{\text{Step 2}}$ Write the equation relating all quantities.

It's the pythagorean theorem, as we just saw:

$$(a(t))^2 + (b(t))^2 = 5^2.$$

$\boxed{\text{Step 3}}$ Implicitly differentiate with respect to t and solve for unknowns

$$\frac{d}{dt}\left((a(t))^2 + (b(t))^2\right) = \frac{d}{dt}(25)$$

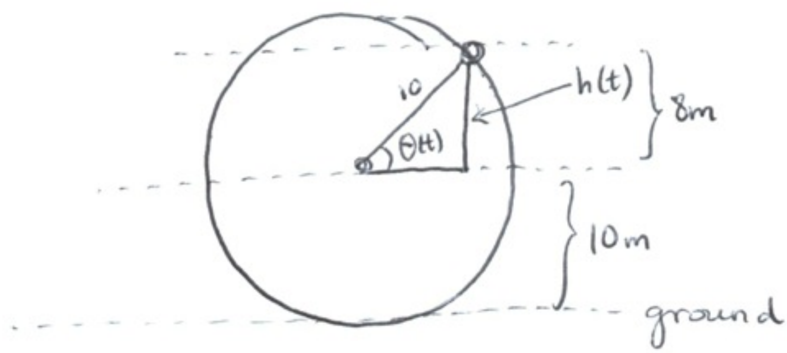
$$\Rightarrow 2a(t)\frac{da}{dt} + 2b(t)\frac{db}{dt} = 0$$

$$\Rightarrow 2 \cdot 3 \cdot \frac{da}{dt} + 2 \cdot 4 \cdot (-1) = 0$$

$$\Rightarrow \frac{da}{dt} = \frac{8}{6} = \frac{4}{3} \text{ m/s}$$

(42) A Ferris wheel with a radius of 10m is rotating at a rate of one revolution every 2 minutes. How fast is the rider rising when their seat is 18m above ground?

Solution: $\boxed{\text{Step 1}}$ Picture and variables



- $h(t)$, the height of the rider above the axis of rotation. We're asked to find $\frac{dh}{dt}$ when $h(t) = 8$m.

- $\theta(t)$, the angle the rider forms with the parallel to the ground. We know $\frac{d\theta}{dt} = \frac{2\pi}{2\text{min}} = \pi/\text{min}$.

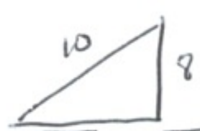
$\boxed{\text{Step 2}}$ Write the equation relating all quantities.

We see that $\sin(\theta(t)) = \frac{\text{opp}}{\text{hyp}} = \frac{h(t)}{10}$,

from this we get that when $h(t) = 8$, $\sin(\theta(t)) = \frac{8}{10}$

so $\theta = \sin^{-1}\left(\frac{4}{5}\right)$. or the

triangle is



$\sqrt{100-64} = \sqrt{36} = 6$

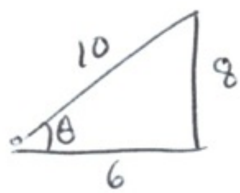
$\boxed{\text{Step 3}}$ Differentiate implicitly and solve for unknowns

$$\frac{d}{dt}\left(\sin(\theta(t))\right) = \frac{d}{dt}\left(\frac{h(t)}{10}\right)$$

$$\Rightarrow \cos(\theta(t))\,\frac{d\theta}{dt} = \frac{1}{10}\,\frac{dh}{dt}.$$

We know $\frac{d\theta}{dt} = 2$. But what is $\cos(\theta(t))$ when $h(t) = 8$?

Recall:



and $\cos = \frac{\text{adj}}{\text{hyp}}$ so $\cos(\theta(t)) = \frac{6}{10}$

$= \frac{3}{5}.$

$$\Rightarrow \left(\frac{3}{5}\right)\cdot(2) = \frac{1}{10}\,\frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{60}{5} = 12\,\text{m}/\text{min}.$$