MATH 1500 Calculus. February 3 **1 not 0**

Lecture 13.

Last day we learned rules for taking derivatives of polynomials and exponentials. Today, two new rules: The product rule and the quotient rule.

<u>Product rule</u>: If $f'(x)$ and $g'(x)$ exist, then

$$(f(x)g(x))' = f'(x)g(x) + g'(x)f(x) \quad \text{or} \quad \frac{d}{dx}(fg) = g\frac{df}{dx} + f\frac{dg}{dx}$$

<u>Example</u>: If $f(x) = e^x(3x^2+x)$, what is $\frac{df}{dx}$?

<u>Solution</u>: We recognize $f(x)$ as a product of two functions e^x and $(3x^2+x)$. We'll call them $h(x) = e^x$ and $g(x) = 3x^2+x$. Then $f(x) = h(x)g(x)$, and the product rule says:

$$\frac{df}{dx} = g\frac{dh}{dx} + h\frac{dg}{dx}.$$

Using the rules from last day:

$$\frac{dh}{dx} = \frac{d}{dx}(e^x) = e^x, \quad \frac{dg}{dx} = \frac{d}{dx}(3x^2+x) = 6x+1.$$

Substituting into the equation for $\frac{df}{dx}$:

$$\frac{df}{dx} = (3x^2+x)e^x + e^x(6x+1)$$

$$= e^x(3x^2+x+6x+1) = e^x(3x^2+7x+1).$$

<u>Example</u>: Calculate the derivative of $(x^2+1)(x^3-x)$ (i) using the product rule, and (ii) by first expanding the expression.

Solution: We have:

Product rule:

$$\begin{aligned}((x^2+1)(x^3-x))' &= (x^2+1)'(x^3-x) + (x^2+1)(x^3-x)' \\ &= 2x(x^3-x) + (x^2+1)(3x^2-1) \\ &= 2x^4 - 2x^2 + 3x^4 - x^2 + 3x^2 - 1 \\ &= 5x^4 - 1\end{aligned}$$

Expanding first:

$$\begin{aligned}(x^2+1)(x^3-x) &= x^5 - x^3 + x^3 - x \\ &= x^5 - x\end{aligned}$$

So the derivative is $(x^5-x)' = 5x^4 - 1$.

So the formula works! But also, it doesn't make things easier in cases where you can just expand the expression.

Why the product formula is true (A.K.A another proof for the midterm!)

Apply the definition:

$$(f(x)g(x))' = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)g(x+h) \overbrace{- f(x+h)g(x) + f(x+h)g(x)}^{\text{This is the essential trick.}} - f(x)g(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x)}{h} + \lim_{h\to 0} \frac{f(x+h)g(x) - f(x)g(x)}{h}.$$

$$= \lim_{h\to 0} f(x+h)\left(\frac{g(x+h)-g(x)}{h}\right) + \lim_{h\to 0} g(x)\left(\frac{f(x+h)-f(x)}{h}\right)$$

$$= \lim_{h\to 0} f(x+h) \lim_{h\to 0}\left(\frac{g(x+h)-g(x)}{h}\right) + \cancel{\lim}\, g(x) \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= f(x)g'(x) + g(x)f'(x).$$

Example: What are the first, second and third derivatives of $f(x) = xe^x$?

Solution: The product rule gives:

$$\frac{df}{dx} = x\frac{d}{dx}(e^x) + e^x\frac{d}{dx}(x) = xe^x + e^x \cdot 1 = e^x(x+1).$$

Then

$$\frac{d^2f}{dx^2} = \frac{d}{dx}\left(e^x(x+1)\right) = e^x\frac{d}{dx}(x+1) + (x+1)\frac{d}{dx}(e^x)$$

$$= e^x \cdot 1 + (x+1)e^x = e^x(x+1) + e^x$$

$$= e^x(x+2).$$

$$\frac{d^3f}{dx^2} = \frac{d}{dx}\left(e^x(x+2)\right) = e^x\frac{d}{dx}(x+2) + (x+2)\frac{d}{dx}(e^x)$$

$$= e^x \cdot 1 + (x+2)e^x = e^x(x+3) \ldots$$

... there is a pattern ...

$$\frac{d^nf}{dx^n} = (x+n)e^x.$$

Quotient Rule: If f and g are differentiable, then

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\frac{df}{dx} - f(x)\frac{dg}{dx}}{(g(x))^2}$$

or $\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$.

Note: Order matters on the top!

Example: Calculate the derivative of $\frac{x^3 + 2x - 1}{x^2}$ using both the product and quotient rule. Verify that they agree.

Solution: If $f(x) = x^3 + 2x - 1$ and $g(x) = x^2$, then the derivative we want is $\frac{d}{dx}\left(\frac{f}{g}\right)$. Since

$\frac{df}{dx} = 3x^2 + 2$ and $\frac{dg}{dx} = 2x$, we get from the quotient rule:

$$\frac{d}{dx}\left(\frac{x^3+2x-1}{x^2}\right) = \frac{x^2(3x^2+2) - (x^3+2x-1)(2x)}{(x^2)^2}$$

$$= \frac{3x^4 + 2x^2 - (2x^4 + 4x^2 - 2x)}{x^4}$$

$$= \frac{x^4 - 2x^2 + 2x}{x^4} = \frac{x^3 - 2x + 2}{x^3}.$$

On the other hand, if $f(x) = x^3 + 2x - 1$ and $g(x) = x^{-2}$, then the derivative we want is $\frac{d}{dx}(fg)$.

Since $\frac{df}{dx} = 3x^2 + 2$, $\frac{dg}{dx} = -2x^{-3}$, we get from the product rule:

$$\frac{d}{dx}(fg) = (x^3 + 2x - 1)(-2x^{-3}) + (3x^2 + 2)(x^{-2}).$$

$$= \frac{(x^3 + 2x - 1)(-2)}{x^3} + \frac{3x^2 + 2}{x^{+2}}$$

$$= \frac{-2x^3 - 4x + 2}{x^3} + \frac{3x^3 + 2x}{x^3}$$

$$= \frac{x^3 - 2x + 2}{x^3}.$$

Example: Find the equation of the line tangent to $y = \frac{e^x}{x}$ at the point $(1, e)$.

Solution: Using the quotient rule, the derivative is:

$$y' = \frac{x(e^x)' - e^x(x)'}{x^2} = \frac{x e^x - e^x \cdot 1}{x^2} = \frac{e^x(x-1)}{x^2}$$

So the slope of the tangent line at $x=1$ is

$$y'(1) = \frac{e^x(1-1)}{\cancel{x}^2 1} = \frac{e^x \cdot 0}{1} = 0.$$

So the tangent line is horizontal. The equation is therefore $y = 0x + b \Rightarrow y = b$, where b is chosen so the line passes through $(1,e)$. Thus $b = e$ and $y = e$.

Example: Calculate $\frac{df}{dx}$ if $f(x) = \frac{e^x}{1+\sqrt{x}}$.

Solution: We need derivatives of top and bottom, so

$$\frac{d}{dx}(e^x) = e^x, \quad \text{and} \quad \frac{d}{dx}(1+\sqrt{x}) = \frac{d}{dx}(1+x^{\frac{1}{2}}) = 0 + \frac{1}{2}x^{-\frac{1}{2}}$$
$$= \frac{1}{2\sqrt{x}}.$$

Then the quotient rule says

$$\frac{df}{dx} = \frac{(1+\sqrt{x})e^x - e^x\left(\frac{1}{2\sqrt{x}}\right)}{(1+\sqrt{x})^2} = \frac{e^x\left(1+\sqrt{x} - \frac{1}{2\sqrt{x}}\right)}{(1+\sqrt{x})^2}$$

$$= \frac{e^x(2\sqrt{x} + 2x - 1)}{2\sqrt{x}(1+\sqrt{x})^2}$$

MATH 1500 Calculus February 5

Lecture 14.

Trig derivatives:

Trig derivatives all depend on the limits

$$\lim_{h\to 0} \frac{\sin(h)}{h} = 1 \quad \text{and} \quad \lim_{h\to 0} \frac{\cos(h)-1}{h} = 0$$

We already saw that

$$\boxed{\frac{d}{dx}(\sin x) = \cos x}$$

in an earlier class, each limit above was needed.

Example: Using the limits above, show $\frac{d}{dx}(\cos x) = -\sin x$.

Solution: If $y=\cos x$, then

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{\cos(x+h)-\cos(x)}{h}$$

$$= \lim_{h\to 0} \frac{\cos(x)\cos(h)-\sin(x)\sin(h)-\cos(x)}{h}$$

$$= \lim_{h\to 0} \frac{\cos(x)\cos(h)-\cos(x)}{h} - \lim_{h\to 0} \frac{\sin(x)\sin(h)}{h}$$

$$= \cos(x) \lim_{h\to 0} \frac{\cos(h)-1}{h} - \sin(x) \lim_{h\to 0} \frac{\sin(h)}{h}$$

$$= \cos(x)\cdot 0 - \sin(x)\cdot 1 = -\sin(x).$$

Example: Calculate $\frac{d}{dx}(\tan(x))$.

Solution: Recall that $\tan(x) = \frac{\sin(x)}{\cos(x)}$.

By the quotient rule,

$$\frac{d}{dx}(\tan(x)) = \frac{\cos x \frac{d}{dx}(\sin x) - \sin x \frac{d}{dx}(\cos x)}{\cos^2(x)}$$

$$= \frac{\cos x \cdot \cos x - \sin(x) \cdot (-\sin(x))}{(\cos(x))^2}$$

$$= \frac{(\cos(x))^2 + (\sin(x))^2}{(\cos(x))^2}$$

$$= \frac{1}{(\cos(x))^2} \text{ or } (\sec(x))^2 \text{ since } \frac{1}{\cos(x)} = \sec(x).$$

We can take derivatives of all the remaining trig functions using the quotient rule and the formulas above. We find:

$$\frac{d}{dx}\sin x = \cos(x) \qquad \frac{d}{dx}\csc(x) = -\csc(x)\cot(x)$$

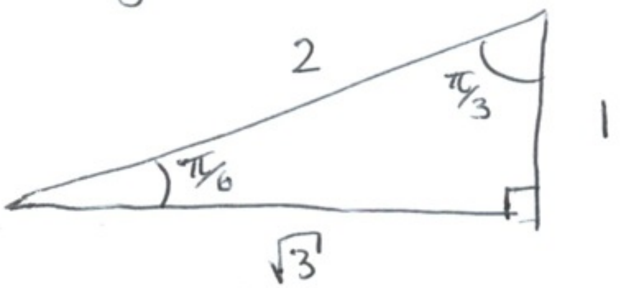
$$\frac{d}{dx}\cos x = -\sin(x) \qquad \frac{d}{dx}\sec(x) = \sec(x)\tan(x)$$

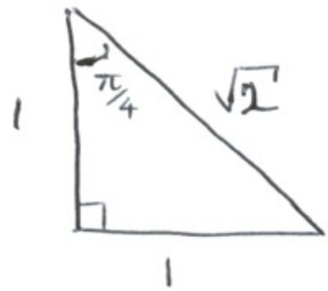
$$\frac{d}{dx}\tan x = \sec^2(x) \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

MEMORIZE THESE!!!

Note: In order to solve ~~these~~ any problems, you have to know how to plug in $0, \pi/6, \pi/4, \pi/3, \pi/2$ into every trig function.

This means you should have memorized the triangles





Then, e.g. $\tan = \frac{\text{opposite}}{\text{adjacent}}$, so $\tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}$.

or $\cos = \frac{\text{adj}}{\text{hyp}}$ so $\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$.

<u>Example</u>: What is the equation of the line tangent to $f(x) = \frac{\sec x}{1+\tan(x)}$ at $x = \pi/4$?

<u>Solution</u>: By the quotient rule:

$$\frac{df}{dx} = \frac{(1+\tan(x))\frac{d}{dx}\sec x - \sec x \frac{d}{dx}(1+\tan(x))}{(1+\tan(x))^2}$$

$$= \frac{(1+\tan(x))\sec x \tan x - \sec x(\sec^2 x)}{(1+\tan(x))^2}$$

$$= \frac{\sec x\left(\tan(x) + (\tan(x))^2 - (\sec(x))^2\right)}{(1+\tan(x))^2}$$

But there is an identity $(\tan(x))^2 - (\sec(x))^2 = -1$, so

$$= \frac{\sec x(\tan(x) - 1)}{(1+\tan(x))^2}.$$

Now we need to find $\left.\frac{df}{dx}\right|_{\pi/4}$, to get the slope of the tangent line.

We need to know $\tan\left(\frac{\pi}{4}\right) = 1$, $\sec\left(\frac{\pi}{4}\right) = \frac{1}{\cos(\pi/4)} = \sqrt{2}$.

$= \frac{hyp}{adj}$.

So the slope of the tangent line is

$$\left.\frac{df}{dx}\right|_{x=\pi/4} = \frac{\sqrt{2}(1-1)}{(1+1)^2} = \frac{\sqrt{2}\cdot 0}{4} = 0,$$ and it

is required to pass through the point $(\pi/4, f(\pi/4))$, which is

$$f(\pi/4) = \frac{\sqrt{2}}{1+1} = \frac{\sec(\pi/4)}{1+\tan(\pi/4)} = \frac{\sqrt{2}}{2}.$$

So the equation of the tangent line is $y = \frac{\sqrt{2}}{2}$.

<u>Example</u>: Differentiate $x^2 \sin x \cos x$.

<u>Solution</u>: Here we have to use the product rule,

$$(fg)' = f'g + g'f.$$

First we set $f(x) = x^2$, $g(x) = \sin(x)\cos(x)$. Then the product rule gives

$$(fg)' = (x^2 \sin x \cos x)' = f'g + g'f$$

$$= (x^2)' \sin x \cos x + (\sin x \cos x)' x^2$$

But the term $(\sin x \cos x)'$ also needs an application of the product rule. We calculate

$$
\begin{aligned}
(\sin x \cos x)' &= (\sin x)' \cos x + (\cos x)' \sin x \\
&= \cos x \cos x + \sin x \sin x \\
&= \cos^2 x - \sin^2 x
\end{aligned}
$$

So the derivative is

$$= 2x \sin x \cos x + x^2(\cos^2 x - \sin^2 x).$$

If you want to practice trig identities, show this is the same as

$$= x(\sin(2x) + x\cos(2x)).$$

<u>Example</u>: What is the second derivative of $\sec(x)$?

<u>Solution</u>: The first is

$$\frac{d}{dx} \sec x = \sec x \tan x$$

then the second is

$$
\begin{aligned}
\frac{d}{dx} \sec x \tan x &= \sec x \frac{d}{dx} \tan x + \tan x \frac{d}{dx} \sec x \\
&= \sec x(\sec^2 x) + \tan x(\sec x \tan x) \\
&= \sec x(\sec^2 x + \tan^2 x).
\end{aligned}
$$

<u>Example</u>: What is the 101st derivative of $f(x) = \cos(x)$?

<u>Solution</u>: Notice that there is a pattern:

$$\frac{d}{dx}\cos(x) = -\sin(x)$$

$$\frac{d^2}{dx^2}\cos(x) = \frac{d}{dx}(-\sin(x)) = -\cos(x)$$

$$\frac{d^3}{dx^3}\cos(x) = \frac{d^2}{dx^2}(-\sin(x)) = \frac{d}{dx}(-\cos(x)) = \sin(x)$$

$$\frac{d^4}{dx^4}\cos(x) = \frac{d^3}{dx^3}(-\sin(x)) = \frac{d^2}{dx^2}(-\cos(x)) = \frac{d}{dx}\sin(x) = \cos(x).$$

And then it repeats: this pattern:

$$-\sin(x), \quad -\cos(x), \quad \sin(x), \quad \cos(x), \ \ldots \text{etc.}$$

1st 2nd 3rd 4th ...

So, eg.

$$\frac{d^{40}}{dx^{40}}(\cos(x)) = \cos(x), \quad \text{and} \quad \frac{d^{100}}{dx^{100}}(\cos(x)) = \cos(x)$$

so $$\frac{d^{101}}{dx^{101}}(\cos(x)) = \frac{d}{dx}\cos(x) = -\sin x.$$

MATH 1500 January 7
Lecture 15

Suggested questions §3.3 Any of 1-16, 19, 25(a), 26(a), 53.
§3.4 1-6, any of 7-46 except 24, 25, 33, 44.

Last day we saw how to use the limits

$$\lim_{h\to 0} \frac{\sin(h)}{h} = 1, \quad \lim_{h\to 0} \frac{\cos(h)-1}{h} = 0$$

to calculate the derivatives of trig functions.
We can also use them to calculate other limits, using substitution / identities.

Example: What is $\displaystyle\lim_{x\to 0} \frac{3\cos(5x)-3}{2x}$?

Solution: Observe that the substitution $x=0$ gives $\frac{0}{0}$, so we need some preliminary algebra.

$$\lim_{x\to 0} \frac{3\cos(5x)-3}{2x} = 3\lim_{x\to 0} \frac{\cos(5x)-1}{2x}$$

$$= 3\lim_{x\to 0} \frac{\cos(5x)-1}{\frac{2}{5}(5x)}$$

$$= 3\lim_{x\to 0} \frac{5}{2}\left(\frac{\cos(5x)-1}{5x}\right)$$

$$= \frac{15}{2}\lim_{x\to 0} \frac{\cos(5x)-1}{5x}$$

However $x\to 0$ means $5x\to 0$, and vice versa. So if we substitute $h=5x$ then the limit becomes $h\to 0$ and:

$$= \frac{15}{2}\lim_{h\to 0} \frac{\cos(h)-1}{h} = \frac{15}{2}\cdot 0 = 0.$$

§ 3.4 The chain rule

Suppose that $f(x)$ and $g(x)$ are differentiable functions. Then the derivative of $(f \circ g)(x) = f(g(x))$ is

$$(f \circ g)' = (f(g(x)))' = f'(g(x)) \cdot g'(x)$$

In Leibniz notation, we write $y = f(u)$ and $u = g(x)$, then $\frac{dy}{dx} = \frac{df}{du} \cdot \frac{du}{dx}$.

Example: If $y = \frac{1}{\sqrt{2x-1}}$, what is $\frac{dy}{dx}$?

Solution: We have to write y as a composition of two functions: $f(u) = \frac{1}{\sqrt{u}}$ and $u(x) = 2x-1$. Then

$y = f(u(x)) = \frac{1}{\sqrt{2x-1}}$. Then in Leibniz notation

$$\frac{df}{du} = \frac{d}{du}\left(u^{-1/2}\right) = -\frac{1}{2}u^{-3/2} = \frac{-1}{2u^{3/2}}$$

and $\frac{du}{dx} = \frac{d}{dx}(2x-1) = 2$.

Then $\frac{dy}{dx} = \frac{df}{du} \cdot \frac{du}{dx} = \frac{-1}{2u^{3/2}} \cdot 2 = \frac{-1}{u^{3/2}}$

However we want our answer in terms of x (as the original question was) so sub in $u = 2x-1$

$$\frac{dy}{dx} = \frac{-1}{(2x-1)^{3/2}}$$

Example: If $y = \sqrt[3]{2+3x}$, what is the equation of the tangent line to this curve at $x=2$?

Solution: We write y as a composition of two functions. If $f(x) = \sqrt[3]{x}$ and $g(x) = 2+3x$, then

$y = f(g(x)) = \sqrt[3]{2+3x}$. The chain rule says

$$y' = (f(g(x)))' = f'(g(x)) \cdot g'(x).$$

We calculate

$$f'(x) = (\sqrt[3]{x})' = (x^{1/3})' = \frac{1}{3} x^{-2/3} = \frac{1}{3\sqrt[3]{x^2}}.$$

should be 3!

Then $g'(x) = 3x$, so

$$y' = f'(g(x)) \cdot g'(x) = \frac{1}{3\sqrt[3]{(2+3x)^2}} \cdot 3x.$$ Therefore the slope of the tangent line at $x=2$ is

$$\frac{1}{3\sqrt[3]{(2+6)^2}} \cdot 3(2) = \frac{6}{3} \frac{1}{\sqrt[3]{64}} = \frac{6}{3} \cdot \frac{1}{4} = \frac{6}{12} = \frac{1}{2}.$$

So the equation is

$y = \frac{1}{2}x + b$, with b so that the line goes through $(2, y(2))$ i.e. $(2, 2)$ $\left(\sqrt[3]{2+3(2)} = \sqrt[3]{8} = 2\right)$.

So

$$2 = \frac{1}{2} 2 + b \Rightarrow b = 2 - 1 = 1.$$

So $y = \frac{1}{2}x + 1$.

Example: What is y' if $y = \cos(e^{x^2})$?

Solution: We need to recognize y as a composition of two functions. If $f(x) = \cos(x)$ and $g(x) = e^{x^2}$, then $y = f(g(x)) = \cos(e^{x^2})$.

So the chain rule gives

$$y' = (f(g(x)))' = f'(g(x))\,g'(x).$$

So we need $f'(x) = -\sin(x)$, and $g'(x) = ??$

In fact g is a composition of two functions!

If $h(x) = e^x$ and $k(x) = x^2$, then

$g(x) = h(k(x)) = e^{x^2}$, so again by the chain rule

So $g'(x) = h'(k(x)) \cdot k'(x)$

and we calculate $h'(x) = e^x$ and $k'(x) = 2x$.

So the formula

$$y' = f'(g(x))\,g'(x) = f'(g(x))\,h'(k(x)) \cdot k'(x) \text{ becomes}$$

$$= -\sin(g(x))\,e^{k(x)} \cdot 2x$$

$$= -\sin(e^{x^2}) \cdot e^{x^2} \cdot 2x.$$

So sometimes, we need two applications of the chain rule.

Example: Find y' and y'' if $y = \cos(x^2)$.

Solution: We write y as a composition: $f(u) = \cos(u)$ and $u(x) = x^2$. Then

$$y' = \frac{dy}{dx} = \frac{df}{du} \cdot \frac{du}{dx}.$$

So we calculate

$$\frac{df}{du} = -\sin(u), \text{ and } \frac{du}{dx} = 2x. \text{ Therefore}$$

$$\frac{dy}{dx} = -\sin(u) \cdot 2x = -\sin(x^2) \cdot 2x.$$

Now we need to use the product rule:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(-\sin(x^2) \cdot 2x\right) = (-\sin(x^2))' \cdot 2x + (-\sin(x^2)) \cdot (2x)'$$

or $2x \dfrac{d}{dx}(-\sin(x^2)) + (-\sin(x^2)) \dfrac{d}{dx}(2x)$

we use the chain rule on the term $\dfrac{d}{dx}(-\sin(x^2))$, as above. We get:

$$\frac{d}{dx}(-\sin(x^2)) = -\cos(x^2) \cdot 2x,$$

so the product rule above gives (after substitution)

$$\frac{d^2y}{dx^2} = 2x(-\cos(x^2)) \cdot 2x + (-\sin(x^2)) \cdot 2$$

$$= -4x^2\cos(x^2) - 2\sin(x^2).$$

<u>Example</u>: We can create new formulas for derivatives by combining the chain rule with old formulas.

For example if n is any real number and $\overset{\frac{dg}{dx}}{\overset{\shortparallel}{g'(x)}}$ exists,

then $$\frac{d(g(x))^n}{dx} = n(g(x))^{n-1} \cdot \frac{dg}{dx}.$$

We can also forget the quotient rule, because combining the product rule and the chain rule works:

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{d}{dx}\left(f(x)\cdot(g(x))^{-1}\right)$$

$$= (g(x))^{-1}\frac{df}{dx} + f(x)\frac{d}{dx}(g(x))^{-1}$$

$$= (g(x))^{-1}\frac{df}{dx} + f(x)\left(-(g(x))^{-2}\right)\frac{dg}{dx}$$

$$= \frac{1}{(g(x))^2}\left[g(x)\frac{df}{dx} - f(x)\frac{dg}{dx}\right] = \text{quotient rule.}$$