Calculus 1500
Lecture 7

January 20

We say that $f(x)$ is continuous at $x=a$ if $\lim_{x\to a} f(x) = f(a)$, and we say that $f(x)$ is continuous on an interval $I=(s,t)$ or $[s,t]$ if f is continuous at every a in I.

Note: If the interval includes the endpoints (e.g $[s,t]$) then we have to use left and right limits to check continuity.

Example: Is $f(x) = \begin{cases} \dfrac{(x+3)(x+1)}{(x+2)} & \text{if } x<0 \\ 3x^2+x+\frac{3}{2} & \text{if } x\geq 0 \end{cases}$

continuous on $[-1,1]$?

Solution: Obvious potential problems: $x=0$ and $x=-2$.
$x=-2$ doesn't matter since we are only asked about the interval $[-1,1]$, which doesn't include $x=-2$.
At $x=0$ we check:

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{(x+3)(x+1)}{(x+2)} = \frac{3\cdot 1}{2} = \frac{3}{2}$$

and $$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} 3x^2+x+\frac{3}{2} = 0+0+\frac{3}{2} = \frac{3}{2}$$

Thus $\lim_{x\to 0} f(x) = \frac{3}{2} = f(0)$. So f is continuous at 0. All other points in $[-1,1]$ are also continuous since f is polynomial/rational and every point of

$[-1,1]$ is contained in the domain of f.

Fact: Adding, subtracting, composing and multiplying continuous functions gives something continuous again.

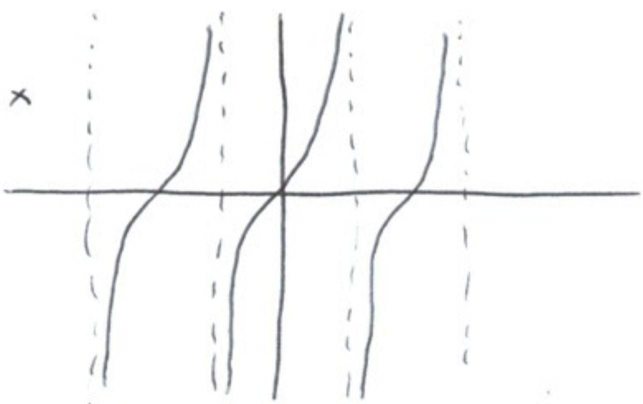
Example: Evaluate $\lim_{x \to 5} \tan\left(\frac{(x+3)(x-5)}{\sqrt{x}}\right)$.

Solution: The function is discontinuous if

- $x < 0$, because then $\sqrt{x}$ is not defined.
- $x = 0$, because then we get division by 0.
- $\frac{(x+3)(x-5)}{\sqrt{x}} = -\frac{3\pi}{2}, \frac{-\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}$, because then we are plugging e.g. $\frac{\pi}{2}$ into $\tan(x) = \frac{\sin(x)}{\cos(x)}$

$$= \frac{1}{0}, \text{ problem.}$$

tan x



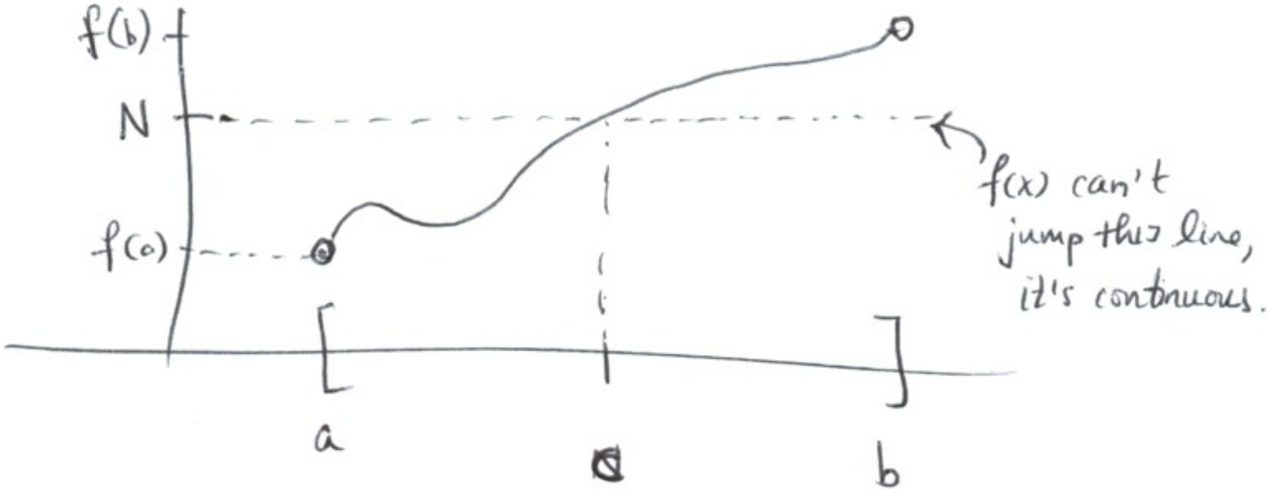
However, none of these problems arise at $x = 5$, where everything is continuous. So $\tan\left(\frac{(x+3)(x-5)}{\sqrt{x}}\right)$ is continuous there and $\lim_{x \to 5} \tan\left(\frac{(x+3)(x-5)}{\sqrt{x}}\right)$

$$= \tan(0) = 0.$$

Theorem (Intermediate value theorem)

Suppose that $f(x)$ is continuous on $[a,b]$, and let N be any number between $f(a)$ and $f(b)$. Then there is a number c in $[a,b]$ so that $f(c) = N$.

E.g.



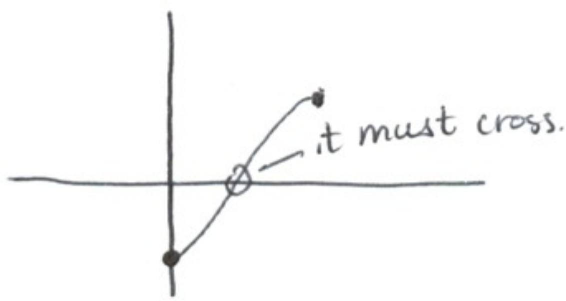
Example: Does $\sin(x) = \cos(x)$ have a solution in $[0, \pi/2]$?

Answer: Maybe you already know that yes, but we can show this using the IV theorem.

Here, $\sin(x) = \cos(x)$ whenever $f(x) = \sin(x) - \cos(x)$ has a zero. Note $f(x)$ is continuous, and

$f(0) = \sin 0 - \cos 0 = -1$, while

$f(\pi/2) = \sin\frac{\pi}{2} - \cos\frac{\pi}{2} = 1$



So by the IV theorem, there is a c with $f(c) = 0$. I.e.

$\sin(c) = \cos(c)$ with $0 \le c \le \frac{\pi}{2}$. In fact $c = \pi/4$.

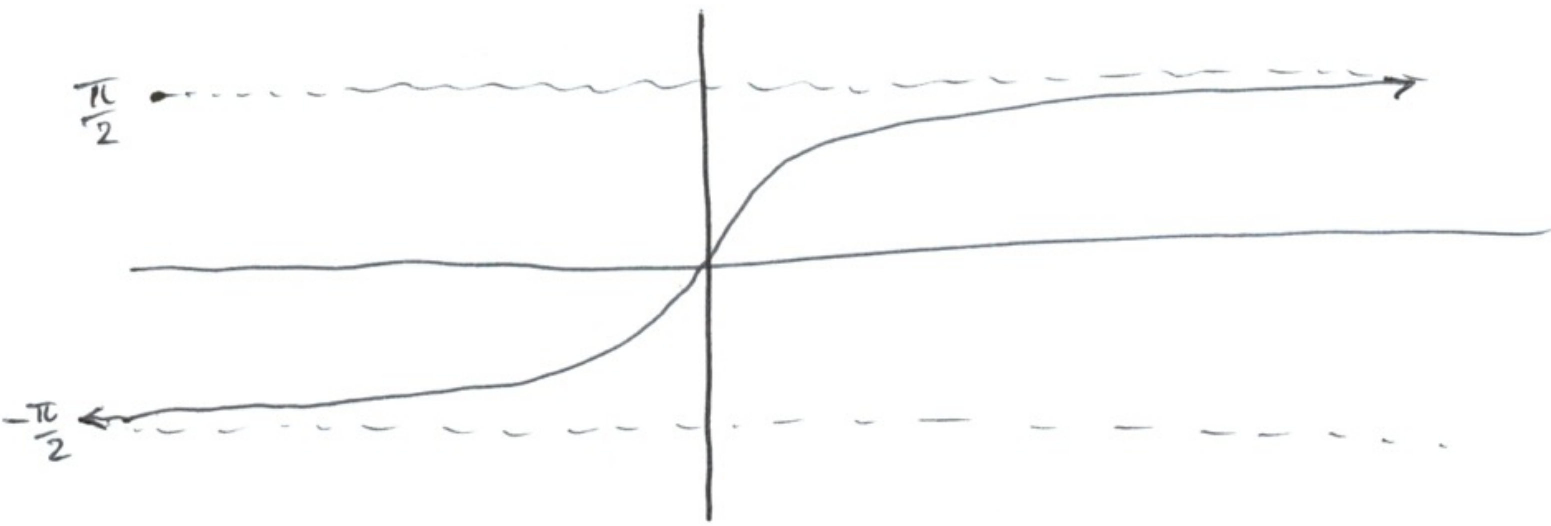
§2.6 Limits at infinity

If I write $\lim_{x \to \infty} f(x) = L$, this means that by taking very, very large numbers c, the value of $f(c)$ can be made as close to L as we like.

Example: $\lim_{x \to \infty} \frac{1}{x} = 0$. Why? Because we can plug very big numbers into $\frac{1}{x}$, and get back numbers as close to 0 as we please.

The graph tends towards the line $y=0$ as $x \to \infty$.

Example: $\lim_{x \to \infty} \tan^{-1}(x) = \frac{\pi}{2}$. The graph of $\tan^{-1}(x)$ is



and we see that as $x \to \infty$, $\tan^{-1}(x)$ tends to the line $y = \frac{\pi}{2}$.

On the other hand, we can take limits as $x \to -\infty$, here we see $\lim_{x \to -\infty} \tan^{-1}(x) = -\infty$.

In general, $\lim_{x \to \pm\infty} \frac{1}{x^n} = 0$

Example (useful trick!).

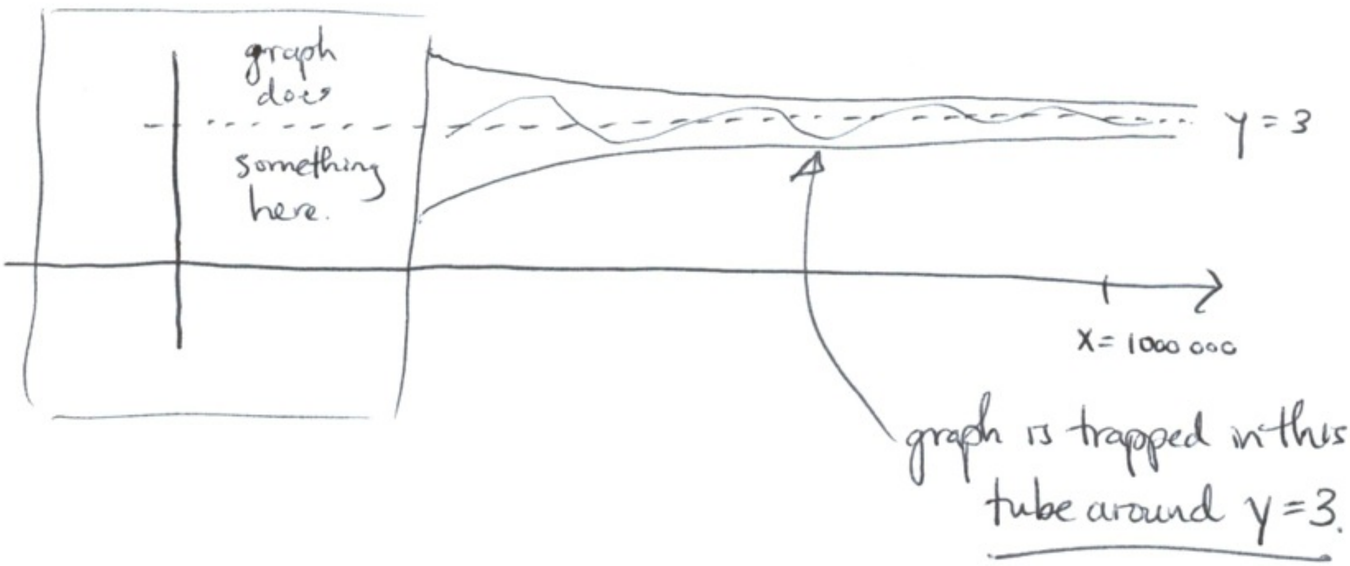
Evaluate $\lim_{x \to \infty} \frac{3x^2 + 2x + 1}{x^2 + 1}$.

Solution: Trick: Multiply both top and bottom by $\frac{1}{x^2}$.

Then
$$\frac{3x^2+2x+1}{x^2+1} = \frac{3\frac{x^2}{x^2} + 2\frac{x}{x^2} + \frac{1}{x^2}}{\frac{x^2}{x^2} + \frac{1}{x^2}}$$
$$= \frac{3 + \frac{2}{x} + \frac{1}{x^2}}{1 + \frac{1}{x^2}}.$$

Then
$$\lim_{x \to \infty} \frac{3x^2+2x+1}{x^2+1}$$
$$= \frac{\lim_{x\to\infty} 3 + \cancelto{0}{\lim_{x\to\infty}\frac{2}{x}} + \cancelto{0}{\lim_{x\to\infty}\frac{1}{x^2}}}{\lim_{x\to\infty} 1 + \cancelto{0}{\lim_{x\to\infty}\frac{1}{x^2}}} = \frac{3}{1} = 3.$$

So the graph of $\dfrac{3x^2+2x+1}{x^2+1}$ looks like:



Terminology:

If $\lim_{x \to \pm\infty} f(x) = L$, the line $y = L$ is a horizontal asymptote of the function $f(x)$.

Example: Calculate $\lim_{x \to \infty} \dfrac{x^2+1}{\sqrt{2x^4+5}}$.

Solution: Multiply the top and bottom by $\frac{1}{x^2}$.

Get: $$\frac{1 + \frac{1}{x^2}}{\frac{1}{x^2}\sqrt{2x^4+5}} = \frac{1 + \frac{1}{x^2}}{\sqrt{\frac{1}{x^4}\left(2x^4+5\right)}}$$

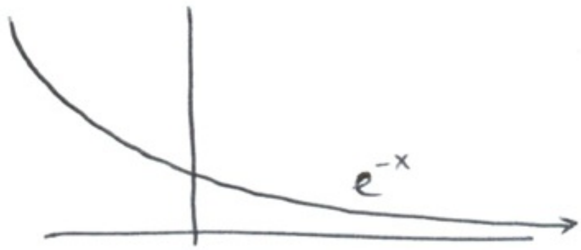
$$= \frac{1 + \frac{1}{x^2}}{\sqrt{2 + 5/x^4}}.$$

So $\lim_{x \to \infty} \dfrac{x^2+1}{\sqrt{2x^4+5}} = \lim_{x} \dots$

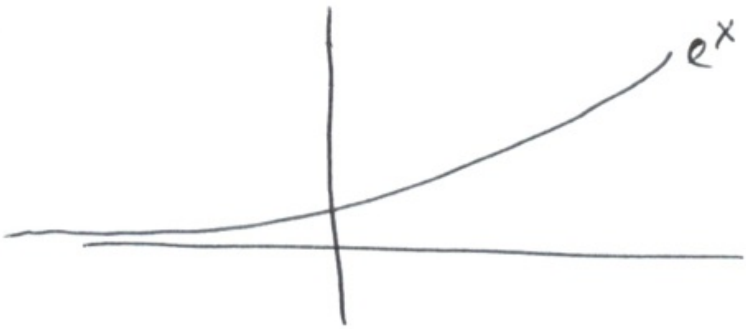
$$= \frac{\lim\limits_{x\to\infty} 1 + \cancel{\lim\limits_{x\to\infty} \frac{1}{x^2}}^{\to 0}}{\sqrt{\lim\limits_{x\to\infty} 2 + \cancel{\lim\limits_{x\to\infty} \frac{5}{x^4}}^{\to 0}}} = \frac{1}{\sqrt{2}}.$$

Additional horizontal asymptotes:

$$\lim_{x\to\infty} e^{-x} = 0$$



$$\lim_{x\to -\infty} e^{x} = 0$$



I.e. $\lim\limits_{x\to\infty} \frac{1}{e^x} = 0$ and $\lim\limits_{x\to -\infty} \frac{1}{e^{-x}} = 0$.

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Lecture 8

Last day, we ended with limits like $\lim_{x\to\infty} f(x) = L$, and $\lim_{x\to-\infty} f(x) = L$. In either case, the horizontal line $y=L$ is called a horizontal asymptote of $f(x)$.

Example: Find the horizontal and vertical asymptotes of $f(x) = \dfrac{\sqrt{5x^2-1}}{x+6}$.

Solution: We find the horizontal asymptotes by taking $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$. Use the trick from last day; dividing by highest power:

$$\lim_{x\to\infty} \frac{\sqrt{5x^2-1}}{x+6} = \lim_{x\to\infty} \frac{\frac{1}{x}\sqrt{5x^2-1}}{\frac{1}{x}(x+6)}$$

$$= \lim_{x\to\infty} \frac{\sqrt{\frac{1}{x^2}}\cdot\sqrt{5x^2-1}}{\frac{x}{x}+\frac{6}{x}}$$

$$= \lim_{x\to\infty} \frac{\sqrt{\frac{1}{x^2}\cdot 5x^2 - \frac{1}{x^2}}}{1+\frac{6}{x}}$$

$$= \lim_{x\to\infty} \frac{\sqrt{5-1/x^2}}{1+6/x}$$

$$= \frac{\sqrt{\lim_{x\to\infty} 5 - \lim_{x\to\infty} \frac{1}{x^2}}}{\lim_{x\to\infty} 1 + \lim_{x\to\infty} 6/x} = \frac{\sqrt{5}}{1} = \sqrt{5}.$$

On the other hand, when we compute the limit as $x \to -\infty$, $\frac{1}{x}$ is $\underline{\underline{\text{negative}}}$ so we can't do the trick of replacing $\frac{1}{x}$ with $\sqrt{\frac{1}{x^2}}$, we need to use $\frac{1}{x}$ replaced with $-\sqrt{\frac{1}{x^2}}$!

So $$\lim_{x\to\infty} \frac{\sqrt{5x^2-1}}{x+6} = \lim_{x\to\infty} \frac{-\sqrt{\frac{1}{x^2}} \cdot \sqrt{5x^2-1}}{\frac{x}{x} + \frac{6}{x}}$$

$= \ldots$ same steps, but with an added minus sign $\ldots$

$= -\sqrt{5}.$

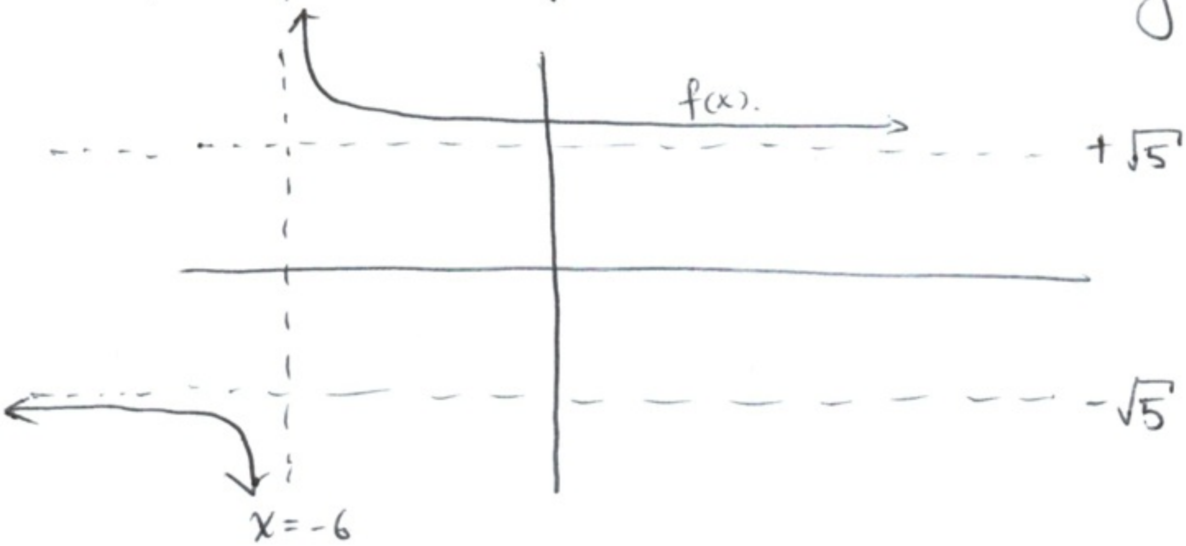
So the horizontal asymptotes of $f(x)$ are $\pm\sqrt{5} = y$.

There is likely a vertical asymptote where the denominator is 0, ie. $x+6=0$ so $x=-6$. Just to be sure:

$$\lim_{x\to -6^+} \frac{\sqrt{5x^2-1}}{x+6} = \frac{(\text{pos})}{(\text{pos})} = +\infty$$

and $$\lim_{x\to -6^-} \frac{\sqrt{5x^2-1}}{x+6} = \frac{(\text{pos})}{(\text{neg})} = -\infty,$$ so yes there's an asymptote.

The function f must look something like:



Example: what is $\lim\limits_{x\to\infty} \dfrac{2e^x-1}{e^x+2}$?

Solution: $\lim\limits_{x\to\infty} \dfrac{1}{e^x} = 0$ so we use the same trick:

$$\lim_{x\to\infty} \frac{2e^x-1}{e^x+2} = \lim_{x\to\infty} \frac{\frac{2e^x}{e^x} - \frac{1}{e^x}}{\frac{e^x}{e^x} + \frac{2}{e^x}} = \lim_{x\to\infty} \frac{2 - \cancel{\frac{1}{e^x}}^{\,0}}{1 + \cancel{\frac{2}{e^x}}^{\,0}}$$

$$= \frac{2}{1} = 2.$$

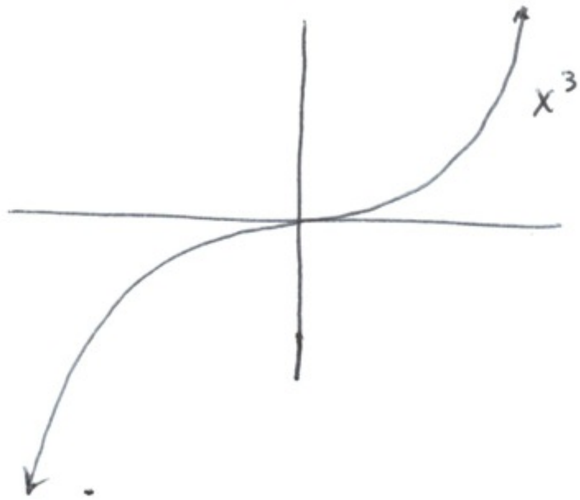
Last, we can also have

$$\lim_{x\to\pm\infty} f(x) = \pm\infty, \text{ or } \lim_{x\to\pm\infty} f(x) = \mp\infty.$$

This means as $x\to\infty$ or $x\to-\infty$, $f(x)$ becomes v. big and pos, or v. big and negative and grows without bound.

__Examples:__ $\lim\limits_{x\to\infty} e^x = \infty$, since

$\lim\limits_{x\to -\infty} x^3 = -\infty$, since



__Example:__ Calculate $\lim\limits_{x\to\infty} \dfrac{x^3+1}{x-5}$.

__Solution:__ Divide by the highest power in the denominator, ie. divide by x. Then

$$\lim_{x\to\infty} \frac{x^3+1}{-x-5} = \lim_{x\to\infty} \frac{\frac{x^3}{x} + \frac{1}{x}}{-\frac{x}{x} - \frac{5}{x}} = \lim_{x\to\infty} \frac{x^2 + \overbrace{\cancel{\tfrac{1}{x}}}^{\to 0}}{-1 - \underbrace{\cancel{\tfrac{5}{x}}}_{\to 0}}$$

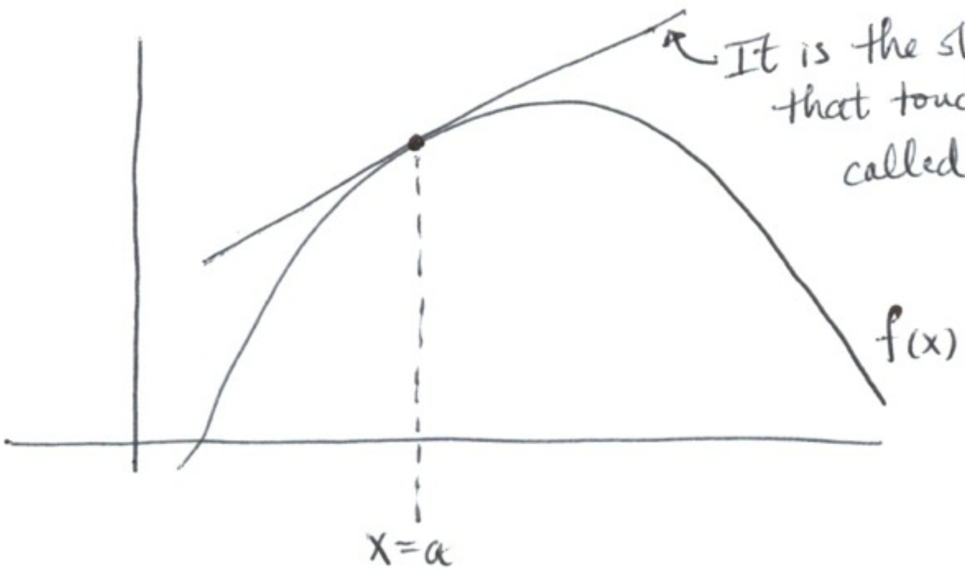
$$= \frac{\lim\limits_{x\to\infty} x^2}{-1} = -\infty.$$

__Warning:__ __Never__ plug ∞ into an equation!

Derivatives, §2.7.

What is the slope of line? Ans: $S = \dfrac{\Delta y}{\Delta x}$.

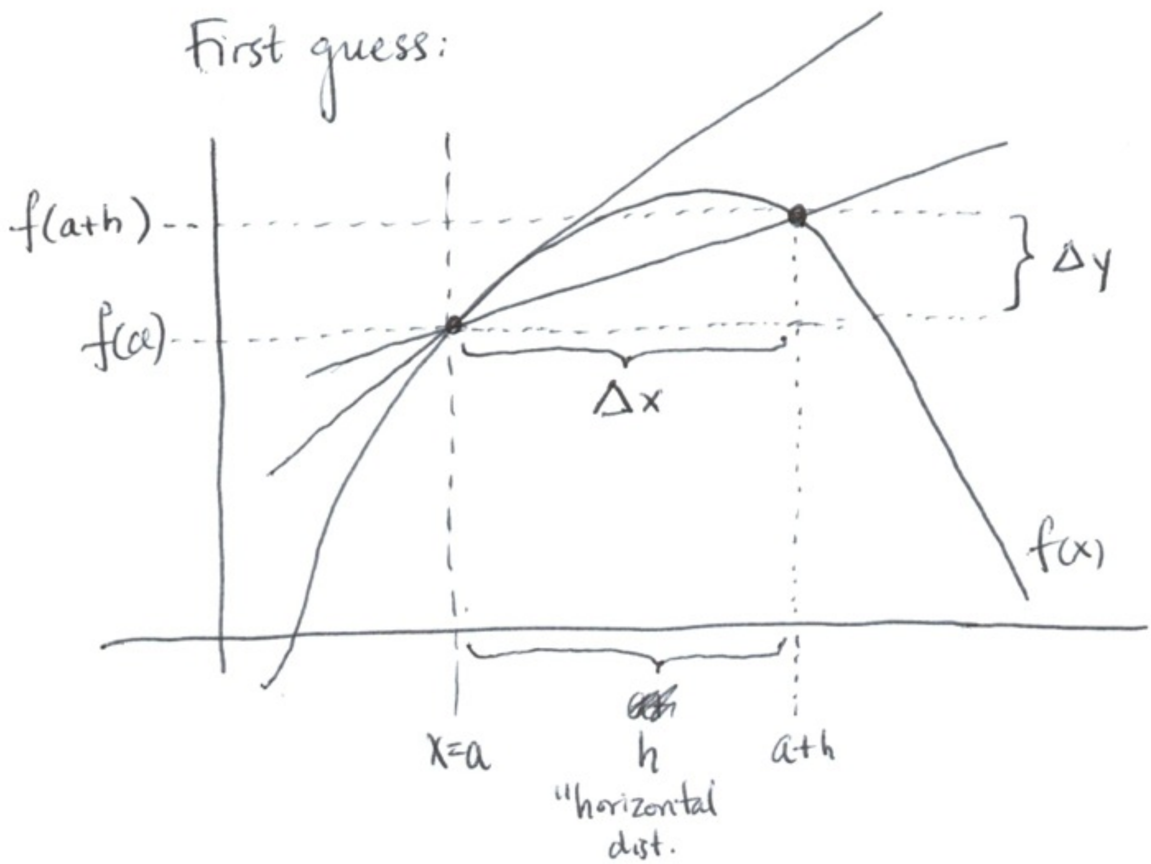
What is the slope of a curve? at $x = a$?



It is the slope of this line that touches f(x) once at x=a, called the tangent line at x=a.

How to calculate the slope of the tangent line?

First guess:



Then the slope is $\dfrac{\Delta y}{\Delta x} = \dfrac{f(a+h)-f(a)}{h}$.

But this is only an approximate slope! Smaller values of h would give better approximations.

Exact value: The slope of $f(x)$ at $x=a$ is the slope of the tangent line at $x=a$. Its slope is $\displaystyle\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$.

Example: What is the slope of the tangent line of $f(x)=x^2$ at $x=3$?

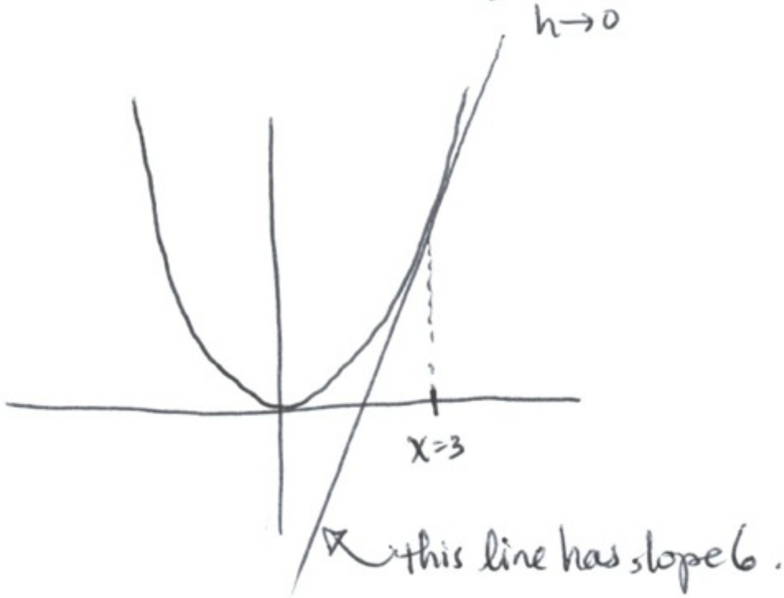
Solution: It is

$$\lim_{h\to 0} \frac{f(3+h)-f(3)}{h} = \lim_{h\to 0} \frac{(3+h)^2-3^2}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{9}+6h+h^2-\cancel{9}}{h}$$

$$= \lim_{h\to 0} 6+h = 6.$$

I.e



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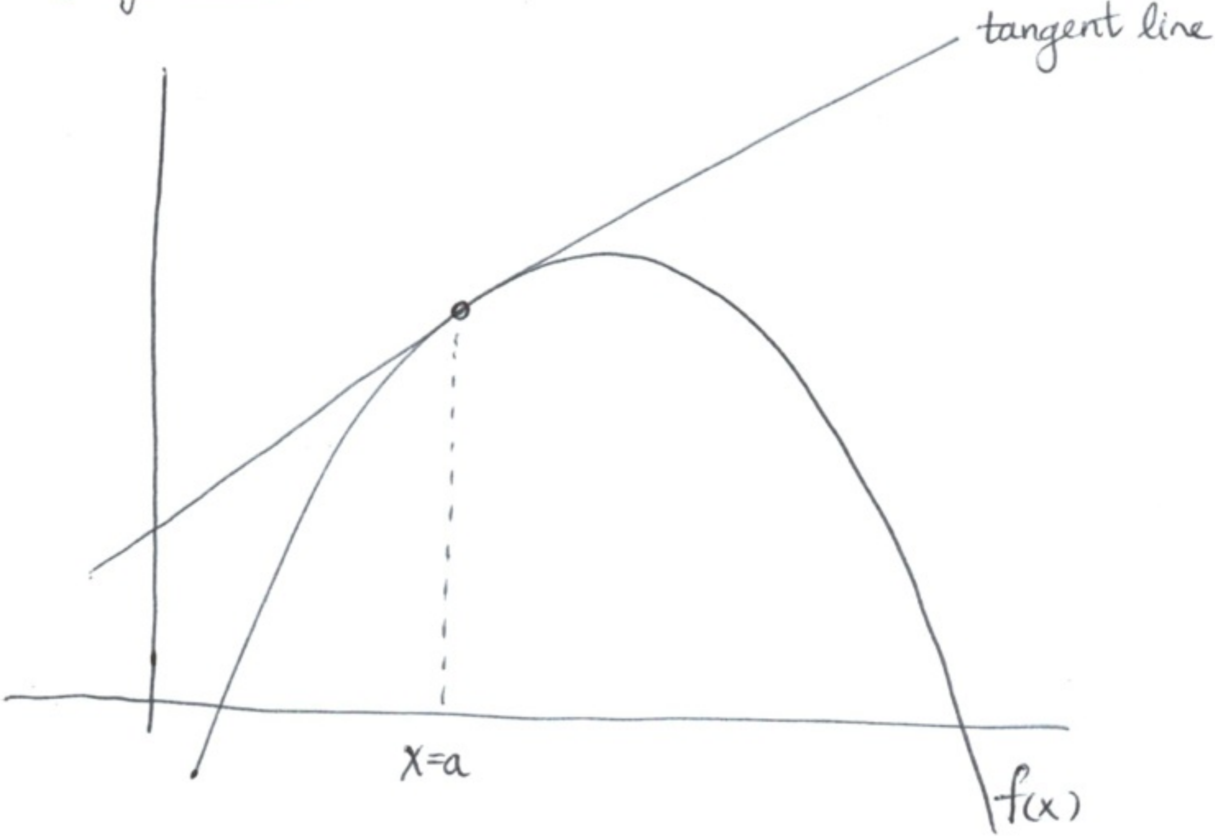
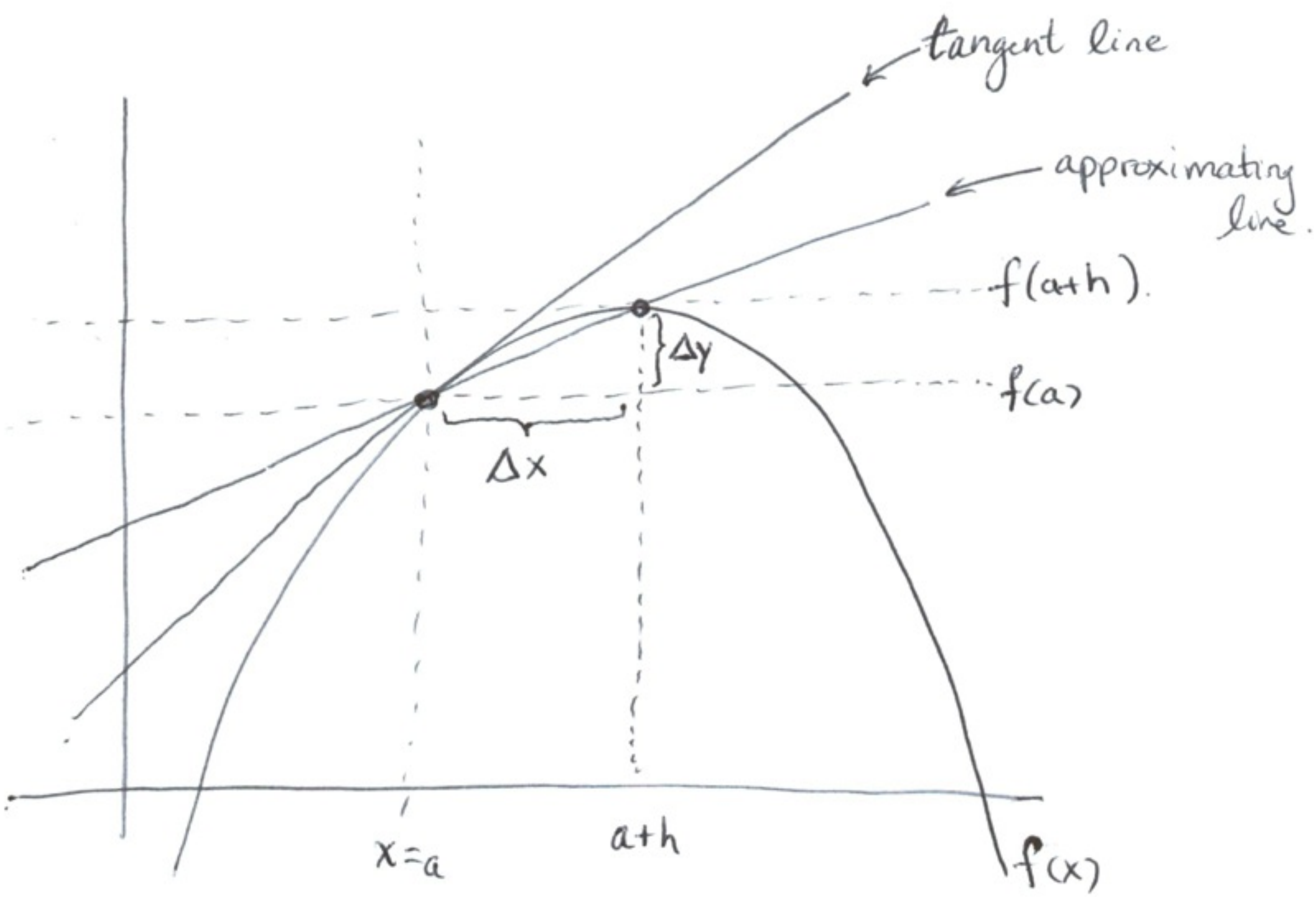
Lecture 9.

Last day we ended with tangent lines:

The slope of $f(x)$ at $x=a$ is the slope of the <u>tangent line</u>.



We can approximate the tangent line with lines that pass through the points $(a, f(a))$ and $(a+h, f(a+h))$. The smaller h is, the better approximation we get:



The slope of the approximating line is

$$\frac{\Delta y}{\Delta x} = \frac{f(a+h)-f(a)}{h}$$

Since the approximation to the tangent line is better when h is smaller, the exact value for the slope of the tangent line is:

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}.$$

Note
This can also be written as

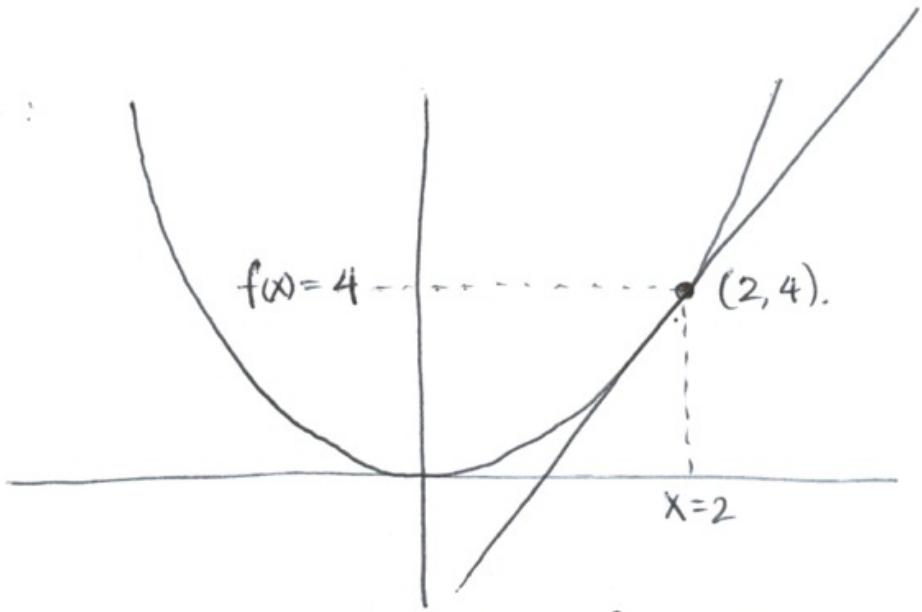
$$\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$

by substituting $h = x-a$ in the first equation. Memorizing either one is fine (but remember at least one of them!)

Example: What is the equation of the tangent line to $f(x) = x^2$ at the point $x = 2$?

Solution:

The picture:



The tangent line passes throught the point (2,4) and its slope is

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \quad , \text{ where } f(x)=x^2 \text{ and } a=2$$

$$= \lim_{h\to 0} \frac{(2+h)^2-(2)^2}{h}$$

$$= \lim_{h\to 0} \frac{4+4h+h^2-4}{h}$$

$$= \lim_{h\to 0} \frac{4h+h^2}{h} = \lim_{h\to 0} 4+h = 4.$$

cancel h's on top and bottom.

So the slope is 4.

Therefore the equation of the tangent line is
$y = mx + b$ where $m = 4$:
$y = 4x + b$ and 'b' is chosen so that the line passes through $(2,4)$:
$4 = 4(2) + b \Rightarrow b = 4 - 8 = -4.$
So the tangent line is $\boxed{y = 4x - 4.}$

The derivative of a function $f(x)$ at the point $x=a$ is written $f'(a)$. The number $f'(a)$ is the slope of the tangent line at $x=a$, in other words:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

Example: What is the slope of the tangent line of $f(x) = x^3$, at an arbitrary point $x=a$?

Solution: The slope formula is

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

$$= \lim_{h \to 0} \frac{(a+h)^3 - a^3}{h}$$

[Expand $(a+h)^3$ and simplify - we did this on day one!

$$= \lim_{h \to 0} \frac{3a^2h + 3ah^2 + h^3}{h}$$

(cancel h's top + bottom)

$$= \lim_{h \to 0} 3a^2 + 3ah + h^2 = 3a^2.$$

So, for example, the slope of the tangent line to $f(x) = x^3$ at the value $x = -6$

is

$$f'(-6) = 3(-6)^2 = 3 \cdot 36 = 108$$

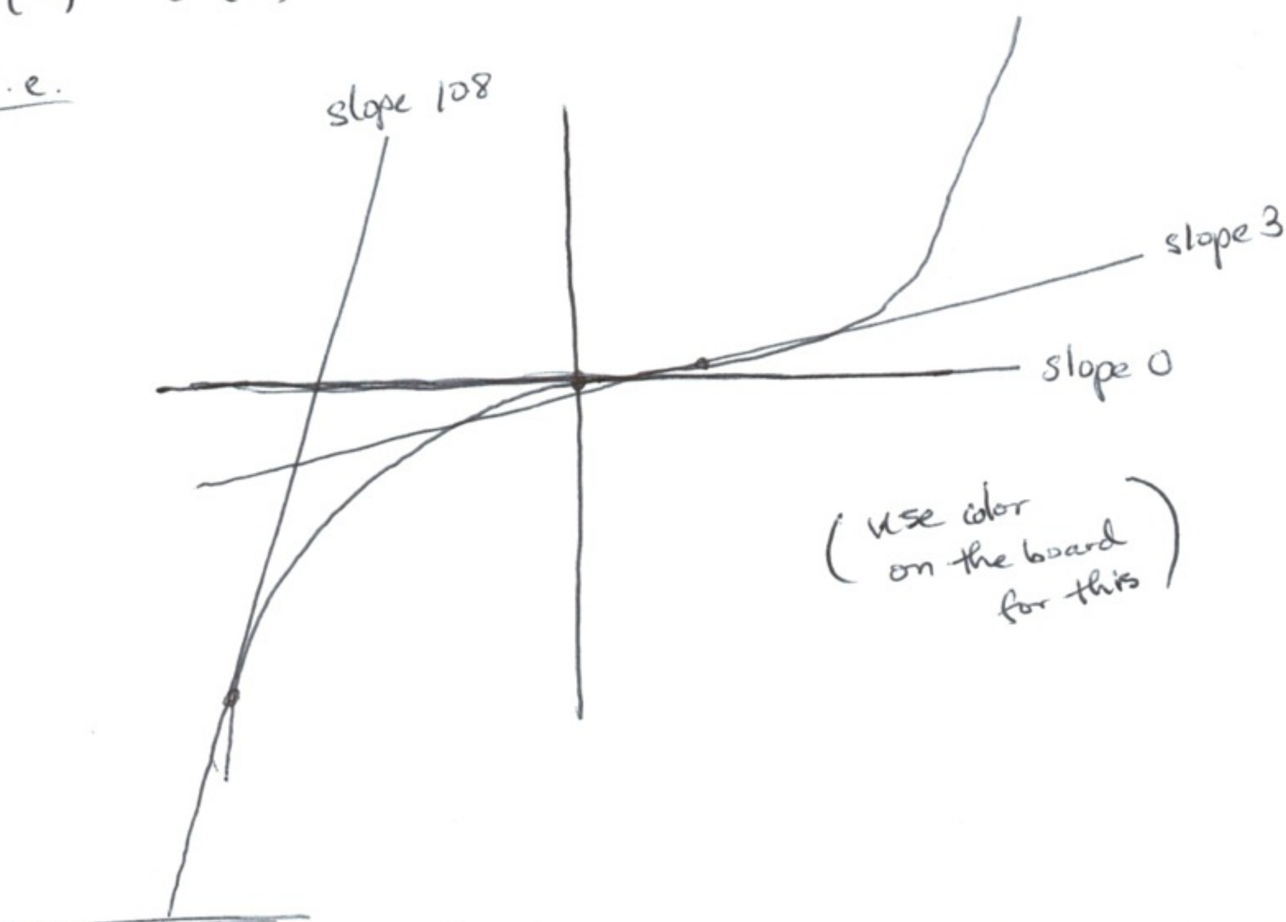
or at $a = 1$ it is

$$f'(1) = 3 \cdot (1)^2 = 3$$

~~ort a~~ or at $a = 0$ it is:

$$f'(0) = 3 \cdot (0)^2 = 0.$$

I.e.



The derivative of $f(x)$ at $x=a$ is also considered as the rate of change of $f(x)$ at $x=a$. (For example, if $f(x)$ is a line then $\frac{\Delta y}{\Delta x}$ is the rate of change of the line)

Or, at $x=a$ $f'(a)$ is called "the instantaneous rate of change of $f(x)$ with respect to x".

Example:

If you drop an object from up high, the distance it travels after t seconds is

$$d(t) = \frac{1}{2}(9.8)\,t^2 \quad \text{(distance in meters).}$$

The (instantaneous) velocity of an object is the rate of change of the distance with respect to time. So the velocity at time $t=a$ is:

$$v(a) = d'(a) = \lim_{h\to 0} \frac{d(a+h)-d(a)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{1}{2}9.8(a+h)^2 - \frac{1}{2}9.8\,a^2}{h}$$

$$= \frac{1}{2}9.8 \lim_{h\to 0} \frac{\cancel{a^2}+2ah+h^2-\cancel{a^2}}{h}$$

$$= \frac{1}{2}9.8 \lim_{h\to 0} 2a+h$$

$$= \frac{1}{\cancel{2}}9.8(\cancel{2}a) = 9.8a.$$

So, for example, after 3 seconds the object is moving at $d'(3) = 9.8(3) = 29.4$ m/s.